Correlation functions between eigenvalues and singular values

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A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy (PhD) of University of Melbourne and KU Leuven

September, 2025

Abstract

Any square complex matrix of size $n \times n$ can be partially characterized by its n eigenvalues and/or n singular values. While no one-to-one correspondence exists between those two kind of values on a deterministic level, for random complex matrices drawn from a bi-unitarily invariant ensemble, a bijection exists between the underlying singular value ensemble and the corresponding eigenvalue ensemble. First, we aim at finding the induced probability measure on j eigenvalues and k singular values that we coin j, k-point correlation measure. We find an expression for the $1, k$-point correlation measure which simplifies drastically when assuming that the singular values follow a polynomial ensemble, yielding a closed formula in terms of the kernel corresponding to the determinantal point process of the singular value statistics. These expressions simplify even further when the singular values are drawn from a Pólya ensemble and extend known results between the eigenvalue and singular value statistics of the corresponding bi-unitarily invariant ensemble.

In a second phase, we focus on polynomial ensembles and derive the large n asymptotics of the $1, k$-point function near the origin (hard edge). An interesting scaling ratio of $1/\sqrt{n}$ is found between the scale of the smallest eigenvalue and the smallest singular value, around the origin. The results are, anew, more explicit for Pólya ensembles, due to their additional integrable structure. This subclass contains all Meijer-G ensembles and, in particular, Muttalib-Borodin ensembles and the classical Wishart-Laguerre (induced Ginibre), Jacobi (truncated unitary), Cauchy-Lorentz ensembles. We show that the latter three ensembles share the same asymptotic of the $1, k$-point function around the origin. In the case of Jacobi ensembles, there exists another hard edge for the singular values, namely the upper edge of their support, which corresponds to a soft edge for the singular value (soft-hard edge). We give the explicit large n asymptotic of the $1, k$-point function around this soft-hard edge.

Keywords: *singular values; eigenvalues; bi-unitarily invariant complex random matrix ensembles; polynomial ensemble; Pólya ensemble;* $1, k$*-point correlation function;* $1, k$*-cross-covariance function; determinantal point process; hard edge scaling*

Declaration

This is to certify that:

1. The thesis comprises only the original work towards the PhD except where indicated below;
2. Due acknowledgement has been made in the text to all other material used;
3. The thesis is fewer than 100 000 words in length, exclusive of tables, maps, bibliographies, and appendices.
4. I did not use generative AI assistance tools during the research/writing process of my thesis, except for mere language assistance.
5. The text/code/images in this thesis are my own (unless otherwise specified) and generative AI has only been used in accordance with the KU Leuven guidelines and appropriate references have been added. I have reviewed and edited the content as needed and I take full responsibility for the content of the thesis.

Chapter 1 is based on the introductions of [8, 10]. Chapter 2 combines part of [8, 10] and paraphrases some parts of [2, 14, 32, 51]. Chapter 3 is based on joint work with Mario Kieburg [10]. Chapter 4 and Chapter 5 are based on [8]. Finally, the concluding Chapter 6 is original. The articles [9] and [11] are not part of the thesis.

Signed,

Matthias Allard

Publications

[1] M. Allard, *Hard edge asymptotics of correlation functions between singular values and one eigenvalue*, Journal of Physics A: Mathematical and Theoretical, **58** (2025), p. 215204, https://doi.org/10.1088/1751-8121/add9f3, https://arxiv.org/abs/2501.15765, [math.PR].

[2] M. Allard, P. J. Forrester, S. Lahiry, and B. Shen, *Partition function of 2D Coulomb gases with radially symmetric potentials and a hard wall*, (2025), https://arxiv.org/abs/2506.14738, [math.PR].

[3] M. Allard and M. Kieburg, *Correlation functions between singular values and eigenvalues*, Random Matrices: Theory and Applications, **15** (2025), p. 2550024, https://doi.org/10.1142/S2010326325500248, https://arxiv.org/abs/2403.19157, [math.PR].

[4] M. Allard and S. Lahiry, *Birth of a gap: Critical phenomena in 2D Coulomb gas*, (2025), https://arxiv.org/abs/2509.24529, [math.PR].

Acknowledgements

I am truly grateful to both of my supervisors, Mario Kieburg and Arno Kuijlaars, for their patience, guidance, and advice. Throughout these four years, they have always been available, and I have learned a great deal from their complementary personalities and backgrounds. I am also grateful to Peter Forrester, with whom collaboration has been a pleasure. He is always supportive and never fails to offer a kind word.

I would like to thank my colleagues and friends—Mathieu Yahiaoui, Sampad Lahiry, and Bojian Shen—for all the time spent discussing and exchanging ideas, both at the office and outside, and for the wonderful collaborative work we have accomplished together, as well as what lies ahead. I am also indebted to all members of both the analysis research group in Leuven and the random matrices research group in Melbourne.

I am grateful to Sung-Soo Byun, Gernot Akemann, and Anas Rahman for inviting me to Jeju, Bielefeld, and Creswick, respectively, and for giving me the opportunity to present my research and engage with the random matrix community.

Finally, I would like to thank my family and friends for putting up with me and continuing to believe in me—despite my arguably questionable life choices, like pursuing an academic career in times like these and relocating to a new country every other year as if I were on a world tour no one asked for.

Preface

Un coup de dés jamais n'abolira le hasard.[1]

Stéphane Mallarmé
(1842-1898)

A fundamental idea in linear algebra is that any linear transformation between finite dimensional spaces can be represented by a matrix. Then, any linear transformation can be thought of as the composition of more elementary linear transformations of two kinds: transformations that preserve the length of objects, called isometries (e.g. rotations, reflections), and transformations that do change the length of objects (e.g. dilations).

Isometries can be thought as encoding a rigid motion: a change of referential which leaves the shape—lengths—of the object unchanged. An object is not necessarily a pencil or an apple but, more generally, an abstract mathematical object such as an element of a vector space. For instance, in quantum physics, a particle is described by a wave function, referred to as state, which encodes its characteristics (momentum, spin, charge, etc). The time evolution of the system—the particle—is given by a linear transformation which acts on the wave function. The objects transformed here are then functions. The linear transformation is called Hamiltonian operator.

The study of random matrices in physics was actually borned out of the study of Hamiltonian operators. In the early 1950's, Eugene Wigner [145, 146] was studying the energy levels of heavy-nuclei atoms and was confronted to not only very complicated Hamiltonians but also to a tremendous number of such Hamiltonians. This is due to the existence of isotopes and the different interactions and coupling between those nuclei, which requires different multi-body Hamiltonians. To manage this complexity, a statistical approach was needed and he proposed replacing the detailed matrix representation of the Hamiltonian with a random matrix that preserved the system's global symmetries. This idea proved to be remarkably successful and laid the foundation for what is now known as random matrix theory (RMT).

On the mathematical side, random matrices were actually introduced in the statistical work of Wishart in 1928 [148]. However, it is with the work of physicists, led by Wigner, Dyson and Mehta who were driven by physical applications, that the mathematical framework of random matrix theory was established in the early

[1] A throw of the dice will never abolish chance.

1960s. First connecting fields of mathematics such as classical analysis and number theory and relying on enumerative combinatorics, more advanced mathematical methods gradually entered the picture: Fredholm determinants (1960s)[2], diffusion processes (1960s), integrable systems (1980s–1990s), and the Riemann–Hilbert problem (1990s), as well as newer approaches like free probability theory (1990s). The diversity of these tools reflects the vitality of the field which develops both on the physics and mathematics side.

One of the key insights in random matrix theory is the observation that the behaviour of large random matrices often exhibits universal properties that are independent of the specific distribution of the matrix entries. These universal properties are themselves shared by systems which are not modelled by random matrices; see Figure 1. Thus, random matrix theory has become a vital tool for understanding complex systems and offers a way to classify seemingly different systems which share the same universal properties, such as the distribution of eigenvalues. Good introductions to random matrix theory and classical results can be found in [2, 14, 32, 51, 102, 115, 136].

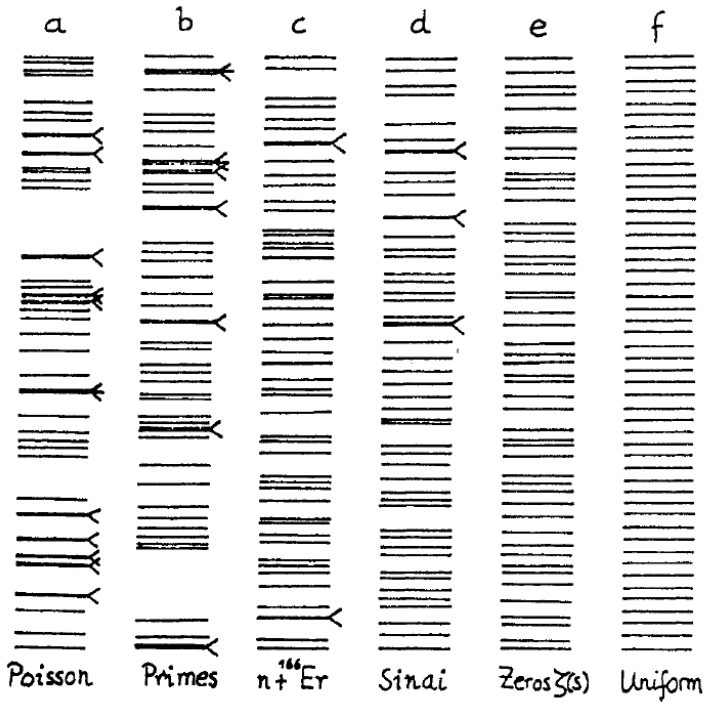

Figure 1: Example of spectra. Some typical spectra of 50 levels, rescaled to the same spectrum span $[0, 49]$. (Reproduced from [21]). **(a)** Random levels with no correlations, Poisson series. **(b)** Sequence of prime numbers. **(c)** Slow neutron resonance levels of the erbium-166 nucleus. **(d)** Possible energy levels of a particle free to move inside the area bounded by $\frac{1}{8}$ of a square and a circular arc whose center is the midpoint of the square; i.e., the area specified by the inequalities $0 \leqslant y \leqslant x \leqslant 1$, and $r^2 \leqslant x^2 + y^2$. (Sinai's billiard table.) **(e)** The zeros of the Riemann zeta function on the line $\mathrm{Re}(z) = \frac{1}{2}$. **(f)** A sequence of equally spaced levels.(Picket fence)
Each arrow represents 2 levels whose distance is less than $1/4$. One can then count the number of arrows in each spectra to find similarities, and, more generally, study the distribution of the level spacings; see [21] for a longer discussion.

[2]Their use precedes their explicit introduction in the 1990s.

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Chapter 1

Introduction

Young man, in mathematics you don't understand things. You just get used to them.

John von Neumann (1903-1957)

1.1 Motivation and state of the art

For general complex square matrices, there exist various different decompositions. We are interested in two in particular, namely the singular value decomposition (SVD) and the Schur decomposition with which we can obtain the eigenvalues of a matrix, i.e.,

(i) *Singular Value Decomposition (SVD):*

$$\forall X \in \mathbb{C}^{n\times n},\ \exists\, \Sigma \in \mathbb{R}_+^n,\ U, V \in \mathrm{U}(n), \quad s.t. \quad X = U\Sigma V \tag{1.1.1}$$

with $\mathrm{U}(n)$ the group of unitary matrices and $\mathbb{R}_+$ the positive real line including 0. With the notation $\mathbb{R}_+^*$ we denote the case when we exclude 0. The matrix Σ is non-negative and diagonal, and its entries are the *singular values* of the matrix X.

(ii) *Schur Decomposition:*

$$\forall X \in \mathbb{C}^{n\times n},\ \exists\, Z \in \mathbb{C}^n,\ T \in \mathrm{T}(n), U \in \mathrm{U}(n), \quad s.t. \quad X = UZTU^\dagger, \tag{1.1.2}$$

where † denotes the Hermitian conjugation and $\mathrm{T}(n)$ the group of upper unitriangular matrices. The matrix Z is complex and diagonal, and its entries are the *eigenvalues* of the matrix X.

The singular value decomposition and Schur decomposition, in general, exhibit ambiguities. Indeed, the singular values $\{\sigma_1, \ldots, \sigma_n\}$ and the moduli of the eigenvalues $\{z_1, \ldots, z_n\}$, which will be called eigenradii $\{|z_1|, \ldots, |z_n|\}$ are not ordered. Moreover, both singular values and eigenvalues are left unchanged under the mapping

$U \mapsto e^{i\theta}U$, and $V \mapsto e^{-i\theta}V$, for $\theta \in \mathbb{R}$. When eigenradii and singular values are pairwise distinct, then, the two decompositions are made unique by replacing $\mathrm{U}(n)$ by the coset space $\mathrm{U}(n)/\mathrm{U}(1)^n$ and ordering eigenradii and singular values.

Remark 1.1.1 *In the case of degeneracies, one needs to consider smaller coset spaces. Namely, for the singular value decomposition* (1.1.1)*, the unitary matrices U and V have to be taken from the coset space*

$$(\mathrm{U}(n) \times \mathrm{U}(n)) \,/\, (\mathcal{Q}_{\mathrm{SV}} \times \mathrm{S}_n)\,, \quad \mathcal{Q}_{\mathrm{SV}} := \{U_1, U_2 \in \mathrm{U}(n) \mid U_1 \Sigma U_2 = \Sigma\}, \tag{1.1.3}$$

with S_n the group of permutation of n elements. For the Schur decomposition (1.1.2) *the unitary matrix U needs to be taken from*

$$\mathrm{U}(n) / \,(\mathcal{Q}_{\mathrm{EV}} \times \mathrm{S}_n)\,, \quad \mathcal{Q}_{\mathrm{EV}} := \{U_1 \in \mathrm{U}(n) \mid U_1 Z T U_1^\dagger = ZT\}. \tag{1.1.4}$$

One needs to resort to the Schur decomposition as not every complex matrix enjoys an eigenvalue decomposition in the form $X = UDU^{-1}$, with D a diagonal matrix and an invertible matrix $U \in \mathrm{GL}(n, \mathbb{C})$. Every linear transformation, represented by a complex matrix $X \in \mathbb{C}^{n\times n}$ can be almost entirely (up to basis transformations) characterised by its singular values $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and its eigenvalues $Z = \mathrm{diag}(z_1, \ldots, z_n)$. Note that in the particular case of Hermitian matrices, i.e. $X = X^\dagger$, the singular values are nothing else than the absolute values of the eigenvalues; $\mathrm{diag}(|z_1|, \ldots, |z_n|) = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. We are, however, interested in non-Hermitian matrices where the relation is not trivial.

Both decompositions enjoy a multitude of applications, usually either the eigenvalues or the singular values alone. While those two sets of values are often studied or used separately depending on the context, they can be complementary. Indeed, the information they capture is different because, albeit being related, there is no one-to-one correspondence between the two sets of values. This is why looking at both eigenvalues and singular values can be relevant and even crucial in certain areas. For instance, in the study of random non-Hermitian Hamiltonians and, in particular, in quantum chaos and open quantum systems [28, 75, 119, 124, 127], the physics and the chaotic behaviour of the system seems to be best captured by the singular values, without making the eigenvalues redundant [16, 126]. The study of non-Hermitian quantum Hamiltonians via the singular values is sometimes referred to as biorthogonal quantum mechanics [27]. The same complementarity is true in Quantum Chromodynamics [87, 88] as well as in topological statistics of Hamiltonians [24, 68–70] or even in Time Series Analysis of time-lagged matrices [20, 103, 105, 121, 141, 152]. It has also become a recent focus of interest in the context of large language models (LLMs) [153].

On a practical level, computing a singular value decomposition is much more stable (cf. [22, Example 1.2]) and faster than an eigenvalue decomposition, hence the use of both sets of values in numerical analysis [13, 33, 84] to improve accuracy and algorithm complexity. Note also that the notion of singular values carries over to general rectangular matrices while eigenvalues are only defined for square matrices.

Many standard results about the statistics of singular values and eigenvalues can be found in [2, 14, 29, 30, 39, 51, 115]. Recent works have looked at the resulting probability density of eigenvalues of products of random matrices e.g. [3–7, 53, 80, 92, 93, 98, 99], sum of random matrices e.g. [97, 120] and also investigated what happens to the distribution of eigenvalues when one would delete columns and rows of the matrix [12, 93].

Despite this broad variety of literature on the subject, singular values and eigenvalues are seldom studied together. The question of their relation is however not new. One of the first results on the subject dates back to 1909 with the work of Schur [131] who proved, what is referred to as Schur's inequality,

$$\sum_{k=1}^{n} |z_k|^2 \leqslant \sum_{k=1}^{n} \sigma_k^2 = \mathrm{Tr}\big(X^\dagger X\big). \tag{1.1.5}$$

This result is immediate when looking at (1.1.1) and (1.1.2). In general, there exists only one equality between singular values and eigenvalues of a matrix $X \in \mathbb{C}^{n\times n}$, which is given by the modulus of the determinant

$$|\det(X)| = |\det(Z)| = \prod_{k=1}^{n} |z_k| = \sqrt{\det(X^\dagger X)} = \det(\Sigma) = \prod_{k=1}^{n} \sigma_k. \tag{1.1.6}$$

However, there exist various inequalities (actually infinitely many of them) such as Weyl's inequalities [144]. After ordering the eigenvalues and singular values like $|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$ and $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n$, Weyl proved that, for any increasing function ϕ such that $x \mapsto \phi(e^x)$ is increasing and convex, with $\phi(0) = \lim_{x\to 0} \phi(x) = 0$,

$$\sum_{k=1}^{m} \phi(|z_k|) \leqslant \sum_{k=1}^{m} \phi(\sigma_k) \quad \text{for any } m \leqslant n. \tag{1.1.7}$$

In particular, the simplest Weyl inequalities read

$$\prod_{k=1}^{m} |z_k| \leqslant \prod_{k=1}^{m} \sigma_k \quad \text{for any } m \leqslant n, \tag{1.1.8}$$

and

$$\sum_{k=1}^{m} |z_k| \leqslant \sum_{k=1}^{m} \sigma_k \quad \text{for any } m \leqslant n. \tag{1.1.9}$$

Two immediate consequences follow from those two inequalities. Firstly, the largest singular value bounds the largest eigenradius from above, which is just the case $m = 1$ of (1.1.8). Secondly, the smallest eigenradius is bounded from below by the smallest singular value as we can apply (1.1.8) for $m = 1$ for the inverse matrix X^{-1} if existent, otherwise there is no non-zero bound. Summarising, we have

$$\sigma_n \leqslant |z_n| \qquad \text{and} \qquad |z_1| \leqslant \sigma_1. \tag{1.1.10}$$

Horn then showed in [77] that given $(z'_1, \ldots, z'_n) \in \mathbb{C}^n$ and $(\sigma'_1, \ldots, \sigma'_n) \in \mathbb{R}^n_+$ with

$|z'_1| \geqslant |z'_2| \geqslant \ldots \geqslant |z'_n|$ and $\sigma'_1 \geqslant \sigma'_2 \geqslant \ldots \geqslant \sigma'_n$, if they verify (1.1.6) and (1.1.9) then they correspond respectively to the eigenvalues and singular values of a complex matrix $X' \in \mathbb{C}^{n\times n}$. As these relations hold for deterministic matrices, they must also hold for random matrices. Therefore, these bounds might be the source of non-trivial correlations between eigenvalues and singular values which may even survive in the limit of large matrix size. The question is then: *On a probabilistic level, what can be said about the relation between eigenvalues and singular values of a complex square matrix?*

A few asymptotic results are known, such as Feinberg-Zee's single ring theorem [48], which has been rigorously proven in [66], for which some hard-to-check condition has then been lifted by [128]. The result of the single ring theorem is two-fold: for a broad class[1] of non-Hermitian random matrices, if the empirical singular-value distributions converge weakly, as $n \to \infty$, to a compactly supported probability measure, then the empirical eigenvalue distributions converge in probability to a (deterministic) probability measure. The support of the limiting probability measure on the eigenvalues is either a single annulus or a disk. The map between the two limiting measures is then given by Haagerup-Larsen's theorem [67] (see [19] for a more comprehensive proof) which derived the relation between those limiting measures with the help of free probability techniques.

For matrices of finite size $n \times n$, the result of Kieburg and Kösters [91] provides a bijection between the joint probability density function of the eigenvalues and the one of the singular values, under some assumptions.

A requirement for all this probabilistic relations has been the bi-unitary invariance of the random matrix ensemble; more details are given in subsection 2.1.2. If a density $f_{\mathrm{BU}} \in \mathrm{L}^1(\mathbb{C}^{n\times n})$ has the property

$$f_{\mathrm{BU}}(U_1 X U_2) = f_{\mathrm{BU}}(X) \quad \text{for all } U_1, U_2 \in \mathrm{U}(n) \text{ and } X \in \mathbb{C}^{n\times n}, \tag{1.1.11}$$

it is then called *bi-unitarily invariant*. This constitutes a large class of matrix ensembles comprising many classical random matrix ensembles. For instance, the induced Ginibre ensemble [2] (also known as Laguerre, Wishart or chiral Gaussian unitary ensemble, when looking at the singular values)

$$f_{\mathrm{Lag}}(X) \propto \det\bigl(X^\dagger X\bigr)^\alpha \exp\bigl(-\operatorname{Tr}\bigl(X^\dagger X\bigr)\bigr),\ \alpha > -1, \tag{1.1.12}$$

as well as the induced Jacobi ensemble [2, 51] (also known as the ensemble of truncated unitary matrices)

$$f_{\mathrm{Jac}}(X) \propto \det\bigl(X^\dagger X\bigr)^\alpha \det\bigl(\mathbb{1}_n - X^\dagger X\bigr)^\beta \Theta(\mathbb{1}_n - X^\dagger X),\ \alpha > -1, \beta > 0, \tag{1.1.13}$$

with $\mathbb{1}_n$ the $n \times n$ identity matrix and Θ the Heaviside step function equal to 1 when the argument is a positive definite matrix and 0 otherwise. Another example is the

[1] Bi-unitarily invariant random matrices; cf. (1.1.11) and (2.1.4).

Cauchy-Lorentz ensemble [147]

$$f_{\mathrm{CL}}(X) \propto \frac{\det\left(X^\dagger X\right)^\alpha}{\det(\mathbb{1}_n + X^\dagger X)^{\beta+2(n-1)}},\ \alpha > -1,\ \beta - \alpha > 1, \tag{1.1.14}$$

but, more generally, any ensemble whose density depends only on $\det\big(P(X^\dagger X)\big)$ and $\mathrm{Tr}\big(Q(X^\dagger X)\big)$, where the functions P, Q are any (convergent) series.[2]

Our aim is to explore more the relations between eigenvalues and singular values for finite matrix size. To do so, we will try to find explicit expression for the joint probability measure of both singular values and eigenvalues and the marginals between j eigenvalues and k singular values; cf. subsection 2.3.3.

The starting point is then to exploit the bijection found in [91]. The related works [89, 93, 94] bring some tools to exploit this bijection when the singular values are drawn from a particular kind of ensembles, such as polynomial ensembles [92, 98, 99] and, more particularly, for Pólya ensembles, which were formerly coined polynomial ensembles of derivative type [60]; more details are given in Chapter 2.

The thesis is organized as follows. In Chapter 2 we give a brief overview of key aspects of random matrix theory and introduce the notations and definitions of the objects we are interested in. We also recall some tools of harmonic analysis. In Chapter 3 we present the result about the j, k-point correlation functions found for finite matrix size. This chapter is based on [10]. In Chapter 4, we look at the infinite size limit and proceed with double scaling limit around the edge at the origin for eigenvalues and singular values. In Chapter 5, for Jacobi ensembles, we look at the double scaling limit of the $1, k$-point correlation functions around the upper edge of the eigenradii and singular value support. Those last two chapters are based on [8]. Finally, some concluding remarks are offered in Chapter 6.

[2] Due to Cayley-Hamilton theorem, it ultimately depends only on the traces of the first n powers of $X^\dagger X$, $\left\{\mathrm{Tr}\left((X^\dagger X)^k\right)\right\}_{k=1}^n$.

Chapter 2

Preliminaries

Shoes are an instrument for walking, maths is an instrument for thinking. You can walk without shoes, but you don't get as far.

Jean-Marie Souriau
(1922-2012)

2.1 General aspects of random matrix theory

2.1.1 Study of ensembles

The study of random matrices straddles the line between mathematics and physics and is multifaceted. It connects many mathematical branches such as algebra, analysis, probability, combinatorics, partial differential equations and even number theory. More than often, same aspects can be approached via different perspectives, using different tools from the aforementioned branches of mathematics.

A key aspect of random matrices, on which we are focusing in this work, is the study of the probability distribution of the random matrix. The choice of a matrix space (usually endowed with the corresponding Borel σ-algebra) and a probability measure on this space is called *random matrix ensemble*.

Random matrix ensembles are broadly divided into two (non mutually exclusive) classes.

- The random matrices built entry wise. When entries are taken to be independent random variables the class is known as *Wigner ensembles*[1], as they have been first introduced by the physicist Eugene Wigner to study the energy levels of heavy nuclei.

[1]We note that Wigner only considered Hermitian matrices with independent entries (up to symmetry) [145, 146]. Girko later considered the non-Hermitian version [63]. For some authors, Wigner ensembles then refers only to the Hermitian case.

- The random matrices enjoying some structural symmetry. More precisely, random matrices whose distribution is invariant under a group action. This class is then called *invariant ensembles*.

This corresponds to two different approaches and perspectives depending on the complex system under study. The first one is bottom up, choosing the entries individually, and comes more from data analysis, statistics, when no global symmetry (invariance) of the matrix is known a priori. The other approach is top down, the matrix as a whole should obey some symmetry, the distribution of the individual entries is then induced. This is usually the approach adopted in the study of quantum Hamiltonians.

Note that, in some cases, those two approaches are equivalent, as some ensembles belong to both classes. Indeed, it is the case for Gaussian ensembles, where the entries are taken to be independent real or complex Gaussian variables. Let us consider a complex matrix $X \in \mathbb{C}^{n\times m}$ distributed according to the following probability measure

$$d\mu(X) = \frac{1}{(2\pi)^{nm}} \exp\left(-\frac{1}{2}\operatorname{Tr}\left(X^\dagger X\right)\right) dX = \prod_{j=1}^{n}\prod_{k=1}^{m} \exp\left(-\frac{1}{2}|X_{jk}|^2\right)\frac{dX_{jk}}{2\pi}, \quad (2.1.1)$$

where dX is the Lebesgue measure on $\mathbb{C}^{n\times m}$ (see Table 2.2), and $X_{j,k}$ denotes the j,k entry of the matrix X, $dX_{j,k} = d\operatorname{Re}\{X_{j,k}\}d\operatorname{Im}\{X_{j,k}\}$. The above probability measure on $\mathbb{C}^{n\times m}$ is called *rectangular complex Ginibre ensemble* and simply *complex Ginibre ensemble* (GinUE) in the case of square matrices $n = m$ [31, 62]. The distribution (2.1.1) illustrates the two different perspectives: one can see the probability measure as being the product of the measures on the individual entries or as the measure on the full matrix itself.

One can then use this random matrix X, as a building block to form other random matrices whose distribution will be induced by (2.1.1). The most classical Gaussian ensembles for both rectangular $n\times m$ matrices and square $n\times n$ matrices thus formed are presented in Table 2.1.

2.1.2 Invariant ensembles

The names of the classical ensembles of Gaussian type in Table 2.1 refer to the invariance of their distribution under the conjugate action of the unitary group $\mathrm{U}(n)$ and the orthogonal group $\mathrm{O}(n)$. A random matrix $X \in \mathbb{C}^{n\times m}$—equivalently its distribution $d\mu$—is said to be:

- *left-unitarily invariant* if

$$d\mu(UX) = d\mu(X), \quad \forall U \in \mathrm{U}(n), \quad (2.1.2)$$

- *right-unitarily invariant* if

$$d\mu(XU) = d\mu(X), \quad \forall U \in \mathrm{U}(m), \quad (2.1.3)$$

Matrix Construction	Name	Description
$X^\dagger X$	complex Wishart/ Laguerre unitary ensemble (LUE)	positive definite Hermitian $m \times m$
$\begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix}$	chiral Gaussian unitary ensemble (χGUE)	chiral Hermitian $(n+m) \times (n+m)$
$W = \frac{1}{2}(X + X^*)$	rectangular real Ginibre ensemble	real $n \times m$
$W^\top W$	real Wishart/ Laguerre orthogonal ensemble (LOE)	positive definite symmetric $m \times m$
$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$	chiral Gaussian orthogonal ensemble (χGOE)	chiral real symmetric $(n+m) \times (n+m)$
$n = m$		
X	complex Ginibre ensemble (GinUE)	complex non-Hermitian $n \times n$
$\frac{1}{2}(X + X^*)$	real Ginibre ensemble (GinOE)	real non-Hermitian $n \times n$
$H = \frac{1}{2}(X + X^\dagger)$	Gaussian unitary ensemble (GUE)	Hermitian $n \times n$
$\frac{1}{2}(H + H^\top)$	Gaussian orthogonal ensemble (GOE)	real symmetric $n \times n$

Table 2.1: Classic Gaussian ensembles. Different ensembles built from a matrix $X \in \mathbb{C}^{n \times m}$ distributed according to (2.1.1).

- *bi-unitarily invariant* if it is both right and left-unitarily invariant, i.e.

$$d\mu(UXV) = d\mu(X), \quad \forall (U, V) \in \mathrm{U}(n) \times \mathrm{U}(m), \tag{2.1.4}$$

- *(conjugate/adjoint) unitarily invariant*, when $n = m$, if

$$d\mu(UXU^{-1}) = d\mu(X), \quad \forall U \in \mathrm{U}(n). \tag{2.1.5}$$

The unitary invariance is closely related to time reversal symmetry in quantum physics and this is why many classical ensembles exhibit this invariance [2, 51, 115]. In this work, however, we will be mainly interested in the bi-unitary invariance, for square matrices.

In general one can also consider the invariance under the action of different groups onto different symmetric spaces. Interesting cases are measures on groups that are invariant under the group action. One of the simplest examples being the Lebesgue measure on $(\mathbb{R}, +)$. This enables to generalize the notion of uniform distribution by assigning an invariant volume to subsets of locally compact topological groups. Such measures are called Haar measures.

Definition 2.1.1 *(*Haar measure.*) Let $(G, \cdot)$ be a locally compact topological group. A positive measure μ which is left-invariant:*

$$\mu(gS) = \mu(S), \quad \forall g \in G, \tag{2.1.6}$$

and all Borel sets $S \subseteq G$, is called a left Haar measure; cf. [36].

Remark 2.1.2 *Similarly, the right Haar measure is invariant under the right action of G. For locally compact group, the left and right Haar measures are unique, up to a multiplicative constant but do not necessarily coincide. For compact groups, right and left Haar measures coincide and can be uniquely specified by adding the normalization condition $\mu(G) = 1$, which makes it a probability measure; cf. [36].*

Example 2.1.3 *Here are some classical examples of Haar measures (up to a multiplicative constant):*

- *For the group $(\mathbb{R}, +)$, the Haar measure is the Lebesgue measure dx.*
- *For the group $(\mathbb{R}_+^*, \times)$, the Haar measure is*

$$d\mu_H(x) = \frac{dx}{x}. \tag{2.1.7}$$

- *For the general linear group $\mathrm{GL}(n, \mathbb{C})$ both right and left Haar measures coincide and the Haar measure is*

$$d\mu_H(X) = \frac{dX}{|\det(X)|^{2n}}. \tag{2.1.8}$$

The notion of invariant measure and invariant volume element can also be approached via differential geometry and is a way to identify the Haar measure explicitly, for Lie groups, as given by some volume form, also called metric form. The metric form then encodes the geometry of the space. More details can be found in [51, 74, 79, 117].

In general, any compact Lie group of matrices endowed with its normalised Haar measure forms an invariant ensemble. From there, some classical ensembles are built by looking at the projections of group elements of Lie groups. For instance, consider the group of unitary matrices $\mathrm{U}(n)$ endowed with its normalized Haar measure,

this ensemble is called *circular unitary ensemble* (CUE). Given a matrix U of the CUE, one can look at the upper left block of size $p \times q$. This forms a new random matrix V whose distribution is induced by the distribution of U. The ensemble thus formed is called truncated unitary matrix ensemble or *Jacobi unitary ensemble* (or complex Jacobi ensemble) (JUE). The probability density distribution of $Y := V^\dagger V$ is proportional to

$$Y \mapsto \det(Y)^{p-q} \det(\mathbb{1}_q - Y)^{n-p-q}, \tag{2.1.9}$$

where $\mathbb{1}_q$ is the $q \times q$ identity matrix; cf. [51, Prop. 3.8.2]. The name Jacobi refers to the Jacobi polynomial involved in the expression of the underlying distribution of the eigenvalues of Y.

2.1.3 Underlying ensembles

From the distribution of the full matrix, one is usually interested in studying the underlying distributions of its eigenvalues and/or singular values, which are also ensembles. Those underlying ensembles can be also thought as point processes in the complex plane and real line, respectively, and studied in themselves. This is the node where most of the different branches and subfields of mathematics involved in study of random matrices meet e.g. potential theory, orthogonal polynomials, Brownian motion.

An important object that is frequently encountered in random matrix theory is the Vandermonde determinant (2.1.10). It arises most frequently from the Jacobian of the change of variables from the matrix entry coordinates to the eigenvalue or singular value coordinates, but also appears in the computation of group integrals via the Weyl character formula; cf. [71, 111].

Notation 2.1.4 *We denote the n-dimensional Vandermonde determinant of an n-dimensional vector $x = (x_1, \dots, x_n)$ by*

$$\Delta_n(x) := \det\left[x_j^{k-1}\right]_{j,k=1}^{n} = \prod_{1 \leqslant j < k \leqslant n} (x_k - x_j), \tag{2.1.10}$$

and take the convention $0^0 := 1$ throughout this work.

Notation 2.1.5 *Abusing notation, we will identify vectors of eigenvalues, squared eigenradii and squared singular values with diagonal matrices.*

Remark 2.1.6 *We will consider the space $\mathrm{GL}(n, \mathbb{C})$ instead of $\mathbb{C}^{n \times n}$; as chosen in [91]. This has no consequence on the level of densities as the event of any singular values/eigenvalues being 0 has Lebesgue measure 0. However, this is a technical detail required to use tools from group and geometric analysis; cf. [74].*

Let us consider a measure μ on $\mathrm{GL}(n, \mathbb{C})$ with density f with respect to the

Lebesgue measure. One can show, using Riesz–Markov–Kakutani representation theorem, that the Lebesgue measure factorizes uniquely according to the decomposition considered:

- For the singular value decomposition of a rectangular $m \times n$ ($m \geqslant n$) matrix $X = U\mathcal{E}V$, with $(U, V) \in \mathrm{U}(m) \times \mathrm{U}(n)$ and $\mathcal{E}$ a diagonal $m \times n$ matrix with diagonal $\Sigma \in \mathbb{R}_+^n$[2],

$$f(X)\, dX \propto \det(\Sigma)^{2(m-n)+1} \Delta_n(\Sigma^2)^2 f(U\mathcal{E}V)\, d\Sigma\, d\mu_H^{(m,n)}(U, V) \tag{2.1.11}$$

  where $d\mu_H^{(m,n)}$ is the normalized Haar measure on $(\mathrm{U}(m) \times \mathrm{U}(n))/\mathrm{U}(1)^n$.

- For the Schur decomposition (1.1.2),

$$f(X)\, dX \propto \left(\prod_{k=1}^{n} |z_k|^{2(n-k)} \right) |\Delta_n(Z)|^2 f(UZTU^{-1})\, dZ\, dT\, d\mu_H^{(n)}(U) \tag{2.1.12}$$

  where z_j are the entries of Z and $d\mu_H^{(n)}$ is the normalized Haar measure on $\mathrm{U}(n)/\mathrm{U}(1)^n$.

In both cases, the prefactors in the RHS correspond to the Jacobian associated with the change of coordinates. In general, this Jacobian can be computed using the wedge product or using the Riemannian volume element, the metric form, from differential geometry [43, 51, 79, 114].

In this work we are interested in invariant matrix ensembles and, in particular, in bi-unitarily invariant ensembles; cf. (2.1.4). In this case, the underlying ensemble on the singular values entirely determines the full matrix ensemble. As there is a one-to-one correspondence between the two, we will consider them as equivalent.

Note that, this can lead to confusions as the underlying ensemble might be known under a different name than the full matrix one. For instance, the induced Ginibre ensemble (1.1.12) is in one-to-one correspondence with its underlying ensemble on the squared singular values being the Laguerre ensemble (2.1.16).

An additional source of confusion comes from the fact that many ensembles on the full matrix space can share the same underlying ensemble on their eigenvalues and singular values; see Example 2.1.7 below.

Example 2.1.7 *Let us consider a matrix H from the χGUE (cf. Table 2.1), it is of the form*

$$H = \begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix} \tag{2.1.13}$$

Its positive eigenvalues are the singular values of X and its negative eigenvalues are also the singular values of X but with a minus sign. The probability density of the

[2] In the rectangular case $m > n$, another possible singular value decomposition is to choose U to be semi-unitary, i.e. U of size $m \times n$ such that $U^\dagger U = \mathbb{1}_n$, and $\mathcal{E} = \Sigma \in \mathbb{R}_+^n$. This is done in [43, 114].

eigenvalues $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ *of* H *is proportional to*

$$\Lambda \mapsto \Delta_n(\Lambda^2)^2 \det(\Lambda)^\alpha \exp\big(-\operatorname{Tr}\big(\Lambda^2\big)\big), \quad \alpha = 2(n-m)+1. \tag{2.1.14}$$

Now let us consider a matrix $W = X^\dagger X$ *from the complex Wishart ensemble (cf. Table 2.1). Its eigenvalues are the squared singular values of* X*. The probability density of the eigenvalues* $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ *of* W *is proportional to*

$$\Lambda \mapsto \Delta_n(\Lambda)^2 \det(\Lambda)^\alpha \exp(-\operatorname{Tr}(\Lambda)), \quad \alpha = n-m. \tag{2.1.15}$$

Therefore, in both cases, the underlying probability density on the squared singular values $a = \operatorname{diag}(a_1, \dots, a_n)$ *is proportional to*

$$a \mapsto \Delta_n(a)^2 \det(a)^\alpha \exp(-\operatorname{Tr}(a)), \quad \alpha = n-m, \tag{2.1.16}$$

which is referred to as Laguerre ensemble*. This density on the squared singular values is also shared by a non-Hermitian ensemble, namely the bi-unitarily invariant ensemble on* $\mathrm{GL}(n, \mathbb{C})$ *given by* (1.1.12) *(induced* GinUE*). The same is true with the Jacobi ensemble, which can be obtain via truncated unitary matrices or via a combination of Wishart matrices [51, Proposition 3.6.1] and for which the probability density on the squared singular values is proportional to*

$$a \mapsto \Delta_n(a)^2 \det(a)^\alpha \det(\mathbb{1}_n - a)^\beta, \quad \alpha, \beta \geqslant 0. \tag{2.1.17}$$

The corresponding bi-unitarily invariant ensemble on $\mathrm{GL}(n, \mathbb{C})$ *is given by* (1.1.13)*.*

On the level of the underlying singular value ensembles we will focus on a particular class of ensembles; the polynomial ensembles, and a subclass; the Pólya ensembles, which are presented in the following section.

Remark 2.1.8 *Let us stress once again that there is a one-to-one correspondence between any bi-unitarily invariant density on* $\mathrm{GL}(n, \mathbb{C})$ *and its underlying probability density on the singular values. Therefore, any ensemble on* $\mathbb{R}_+^n$ *(the singular values) is thought as the underlying ensemble of bi-unitarily invariant ensemble on* $\mathrm{GL}(n, \mathbb{C})$*. The designation of a particular class of ensembles on* $\mathbb{R}_+^n$ *is then carried over to* $\mathrm{GL}(n, \mathbb{C})$*. This justifies the formulation "singular values of a polynomial ensemble", which has to be understood as "bi-unitarily invariant ensemble whose underlying ensemble on the singular values is a polynomial ensemble"; more details in subsection 2.3.2.*

2.1.4 Polynomial ensembles

As already mentioned, bi-unitarily invariant ensembles are entirely characterized by the underlying density of the singular values. Thus, in order to find explicit and useful formulas for the correlation functions between eigenvalues and singular values, one has to choose explicit probability density functions on the singular values

f_{SV}. A computationally suitable and rather broad class of ensembles are *polynomial ensembles* [60, 92, 97–99], which are themselves a subclass of *biorthogonal ensembles* [23].

Definition 2.1.9 (Polynomial ensemble) *A polynomial ensemble on $\mathbb{R}_+^n$ is a probability density function of the form*

$$f_{\mathrm{SV}}(x_1, \dots, x_n) = C_{\mathrm{SV}} \Delta_n(x) \det\left[w_{k-1}(x_j)\right]_{j,k=1}^n \geqslant 0 \tag{2.1.18}$$

with the normalisation constant

$$\frac{1}{C_{\mathrm{SV}}} = n! \det\left[\int_0^\infty x^{j-1} w_{k-1}(x) dx\right]_{j,k=1}^n \in \mathbb{R} \setminus \{0\} \tag{2.1.19}$$

and $\{w_b\}_{b=0}^{n-1}$, $w_b \in \mathrm{L}^1(\mathbb{R}_+)$ whose first $n-1$ moments exist.

Note that the structure of polynomial ensembles (2.1.18) appears very naturally in the study of random matrices as the Jacobian of the change of coordinates to go on the singular values yields a squared Vandermonde determinant (cf. (2.1.11)). The examples of the Laguerre and Jacobi ensembles mentioned in Example 2.1.7 belong to this class.

Polynomial ensembles enjoy some additional structure from their belonging to a much larger class of ensembles called *determinantal point processes* (DPP); see [2, 14, 39, 51, 115].

Definition 2.1.10 (Determinantal point process (DPP)) *A simple[a] point process $\mathscr{X}$ on $\mathbb{R}_+$ is said to be a determinantal process with kernel K if the correlation functions R_k satisfy*

$$R_k(x_1, \dots, x_k) = \det\left[K(x_i, x_j)\right]_{i,j=1}^k, \tag{2.1.20}$$

for every $k \geqslant 1$ and $x_1, \dots, x_k \in \mathbb{R}_+$; see subsection 2.3.3 and Remark 2.3.9.

[a] The process is almost surely a discrete set of distinct points.

Remark 2.1.11 *Note that the definition does not assume the existence of a joint probability density function as the point process can have a varying or infinite total number of points. Moreover, the correlation functions R_k are not necessarily marginals of each other, i.e. R_{k-1} is not necessarily obtained by integrating over one of the argument of R_k. This is however the case whenever the associated integral kernel operator $\mathcal{K}$ on $\mathrm{L}^2(\mathbb{R}_+, \mu)$,*

$$(\mathcal{K}f)(x) = \int_{\mathbb{R}_+} K(x, y) f(y) \, d\mu(y), \tag{2.1.21}$$

is a projection operator, which is equivalent to having the reproducing property of

the kernel

$$\int_{\mathbb{R}_+} K(x,t)K(t,y)\,d\mu(t) = K(x,y), \quad \forall x,y \in \mathbb{R}_+. \tag{2.1.22}$$

We refer the reader to [115, Chapter 5] and to [78] for further details.

In our case, the total number of points, n, is fixed and finite. Thus, the joint probability distribution f_{SV} of any polynomial ensemble can be written in the form

$$f_{\mathrm{SV}}(x_1,\dots,x_n) = \frac{1}{n!}\det\left[K_n(x_j,x_k)\right]_{j,k=1}^{n}, \tag{2.1.23}$$

where K_n is the kernel function, and all the k-point correlation functions have a similar form where only the size of the determinant changes. Note that K_n is not uniquely given. Indeed, due to elementary properties of the determinant, for a non-vanishing function g, the kernel $[g(x_1)/g(x_2)]K_n(x_1,x_2)$ is also a correlation kernel for the same point process. However, for polynomial ensemble one can choose K_n to be polynomial of degree $n-1$ in the second entry, which thus makes it unique. Explicitly, K_n can be written as

$$K_n(x,y) = \sum_{b=0}^{n-1} q_b(x)p_b(y), \tag{2.1.24}$$

with p_b a polynomial of degree b and $q_b \in \mathrm{Span}\{w_0,\dots,w_{n-1}\}$ where Span is the linear span. The two sets of functions $\{q_b\}_{b=0,\dots,n-1}$ and $\{p_b\}_{b=0,\dots,n-1}$ must satisfy the bi-orthonormality relation

$$\int_0^\infty q_b(x)p_c(x)dx = \delta_{c,b}, \tag{2.1.25}$$

where $\delta_{c,b}$ is the Kronecker delta function.

Remark 2.1.12 *Note that this does not uniquely define the functions p_j and q_j as one can always decompose the polynomials p_j in another basis of polynomial up to degree $n-1$ and then take the corresponding linear combination of q_j to form a new bi-orthonormal system. In particular, given a sequence of non zero real numbers $\{c_j\}_{j=0,\dots,n-1}$, $\{c_jq_j\}_{j=0,\dots,n-1}$ and $\{c_j^{-1}p_j\}_{j=0,\dots,n-1}$ are also possible bi-orthonormal families. Thus, the p_j are not canonically associated to the ensemble. Note also that p_j, q_j might depend on n, which is not reflected in the notation.*

The 1-point function for a polynomial ensemble, and more generally for any determinantal point process can be expressed in terms of this kernel. It has the rather simple form

$$f_1(x) = \frac{1}{n}K_n(x,x). \tag{2.1.26}$$

It is important to stress that not only does the kernel play a crucial role in the

point process of the singular values but also in the point process of the eigenradii and, a fortiori, in the combined point process as it will be seen in the main results Theorem 3.3.7.

2.1.5 Pólya ensembles

A very important subclass of polynomial ensemble, which enjoys some closure property, is the class of *Pólya ensembles* (formerly coined polynomial ensembles of derivative type) [60, 89, 91–94]. Note that, as can be seen in [60, 94], different definitions of Pólya ensembles are possible depending on the full matrix space under consideration. As we focus, in this work, on complex $n \times n$ matrices, we define a Pólya ensemble as follows—sometimes referred to as Pólya ensemble of multiplicative type [149].

Definition 2.1.13 (Pólya ensemble) *A Pólya ensemble is a polynomial ensemble for which there exists a non-negative $(n-1)$-times continuous differentiable weight function $w_n \in \mathrm{C}^{n-1}(\mathbb{R}_+)$, such that*

$$\mathrm{Span}\{w_{k-1}\}_{k=1}^{n} = \mathrm{Span}\{x \mapsto (-x\partial_x)^{k-1} w_n(x)\}_{k=1}^{n} \tag{2.1.27}$$

and

$$x \mapsto x^j (-x\partial_x)^k w_n(x) \in \mathrm{L}^1(\mathbb{R}_+), \quad \forall j,k \in [\![0, n-1]\!]. \tag{2.1.28}$$

This choice comes very naturally when considering the identity

$$\mathcal{M}[(-x\partial_x)^n f(x)](s) = s^n \mathcal{M} f(s), \quad n \in \mathbb{N} \tag{2.1.29}$$

for the Mellin transform (2.3.1) and the fact that the Mellin transform plays an important role in the relation between the joint probability density functions of the singular values and the eigenvalues.

To guarantee that we deal with probability measures it has been shown in [60] that w_n is then related to Pólya frequency functions, themselves related to totally positive functions [65, 130].

Definition 2.1.14 (Pólya frequency functions) *Let $N \in \mathbb{N}$. A measurable function f is called a Pólya frequency function of order N if*

$$\Delta_n(x)\Delta_n(y) \det\left[f(x_j - y_k)\right]_{j,k=1}^{n} \geqslant 0 \tag{2.1.30}$$

for all $n = 1, \ldots, N$ and all $x, y \in \mathbb{R}^n$. If (2.1.30) hold for any integer $n \in \mathbb{N}$, f is called a Pólya frequency function of infinite order.

Theorem 2.1.15 ([60, Theorem 2.9]) *Let $w_n \in \mathrm{C}^{n-1}(\mathbb{R}_+)$ with $w_n \neq 0$ and such that, for all $j,k \in [\![0,n-1]\!]$, $x \mapsto x^j(-x\partial_x)^k w_n(x) \in \mathrm{L}^1(\mathbb{R}_+)$. Then, w_n gives rise to a Pólya ensemble if and only if $x \mapsto e^{-x}w_n(e^{-x})$ is a Pólya frequency function of order n.*

The key features of Pólya ensembles are the following properties (cf. [60, 92, 97, 98]). Let X and Y be two independent bi-unitarily invariant random matrices whose squared singular values are respectively distributed according to different (or same) Pólya ensembles with weights w_X and w_Y:

1. (*Composition closure*) The matrix XY is bi-unitarily invariant and its squared singular values follow a Pólya ensemble, whose corresponding Pólya weight w_{XY} is given by the multiplicative convolution (2.3.4)
$$w_{XY}(x) = (w_X * w_Y)(x) := \int_0^\infty w_X\left(\frac{x}{y}\right) w_Y(y)\,\frac{dy}{y}. \tag{2.1.31}$$
2. (*Inverse closure*) The random matrix Y^{-1} is bi-unitarily invariant and its singular values follow a Pólya ensemble, whose corresponding Pólya weight $w_{Y^{-1}}$ is given by
$$w_{Y^{-1}}(x) = \frac{1}{x^{n+1}} w_Y\left(\frac{1}{x}\right). \tag{2.1.32}$$
3. (*Determinantal*) The eigenvalues of X follow a determinantal point process on $\mathbb{C}^n$; see (2.1.43).

Remark 2.1.16 *The inverse closure is inherited—or shared, depending on the perspective—from the bi-unitary invariance. Indeed, given $X \in \mathrm{GL}(n,\mathbb{C})$ following a bi-unitarily invariant distribution with density f_{BU}, the distribution of X^{-1} is also bi-unitarily invariant, with density $X \mapsto f_{\mathrm{BU}}(X^{-1})\det(X)^{-4n}$; cf. [92, Remark 3.5].*

The closure property and the explicit formula for the distribution of the composition make the study of this class of ensembles all the more interesting. Moreover, many classical ensembles are Pólya ensembles. It is, for instance, the case of the Laguerre (1.1.12), Jacobi (1.1.13) and Cauchy-Lorentz (1.1.14) ensembles, whose respective corresponding Pólya weight functions (2.1.27) are
$$w_{\mathrm{Lag}}(x) = x^\alpha e^{-x}, \quad w_{\mathrm{Jac}}(x) = x^\alpha(1-x)^{\beta+n-1}\Theta(1-x), \quad w_{\mathrm{CL}}(x) = \frac{x^\alpha}{(1+x)^{\beta+n-1}}. \tag{2.1.33}$$
Those ensembles are well studied and appear in many different applications such as the study of wireless communication systems, entanglement of a random pure quantum state, quantum conductance; see [2, 51, 115]. Note that the Cauchy-Lorentz ensemble is related to the circular unitary ensemble (CUE)—and its trun-

cated version—via a Cayley transformation. On the level of eigenvalues it corresponds to a simple change of variable, namely a stereographic projection from the unit circle to the real line (see [51, p.68]).

Other important ensembles falling in the class of Pólya ensemble are the Muttalib-Borodin ensembles [23, 59] which have been introduced to model two-body interactions in random matrix models [118] and used, for instance, in the study of disordered conductors, quantum transport, Brownian motion [18, 109, 135]. We give the example of the Laguerre type Muttalib-Borodin ensemble whose Pólya weight is given by

$$w_{\mathrm{MB}}(x) = x^{\alpha} e^{-x^{\theta}}, \quad \alpha > -1, \theta > 0. \tag{2.1.34}$$

In fact, not only are most of the classical random matrix ensembles Pólya ensembles but also their composition due to the nice closure properties presented above.

The joint probability density function of a Pólya ensemble has the form

$$f_{\mathrm{SV}}(x_1, \dots, x_n) = C_{\mathrm{SV}} \Delta_n(x) \det\left[(-x_j \partial_{x_j})^{k-1} w_n(x_j)\right]_{j,k=1}^{n}, \tag{2.1.35}$$

with

$$C_{\mathrm{SV}} = \frac{1}{\prod_{j=1}^{n} j! \, \mathcal{M}w_n(j)}. \tag{2.1.36}$$

The kernel, then, admits an integral representation, see [91, Eq.(4.22)] and [98],

$$K_n(x,y) = \sum_{j=0}^{n-1} q_j(x) p_j(y) = n \int_0^1 dt \, q_n(xt) p_{n-1}(yt) \tag{2.1.37}$$

with the bi-orthonormal sets of functions $\{q_j\}_{j=0}^{n-1}$ and $\{p_j\}_{j=0}^{n-1}$ given by

$$p_j(x) = \sum_{c=0}^{j} \binom{j}{c} \frac{(-x)^c}{\mathcal{M}w_n(c+1)}, \quad q_j(x) = \frac{1}{j!} \partial_x^j [x^j w_n(x)], \tag{2.1.38}$$

where we remind this choice is not unique, cf. Remark 2.1.12. For this result we actually need the n-times differentiability of w_n, i.e., $w_n \in \mathrm{C}^n(\mathbb{R})$, to have a well-defined function q_n. If the weight is only $(n-1)$-differentiable one needs to modify the formula to

$$K_n(x,y) = n \int_0^1 dt \, q_{n-1}(xt) p_{n-2}(yt) + q_{n-1}(x) p_{n-1}(y). \tag{2.1.39}$$

Note that the q_j can also be expressed in the following integral form [91, (4.18)],

$$q_j(x) = \lim_{\varepsilon \to 0} \int_{\mathcal{C}_1} \frac{ds}{2\pi i} \zeta(\varepsilon \operatorname{Im}\{s\}) \frac{\Gamma(j+1-s)}{\Gamma(j+1)\Gamma(1-s)} \mathcal{M}w_n(s) x^{-s}, \tag{2.1.40}$$

using the inverse Mellin tranform; see subsection 2.3.2, which involves the regularisation function ζ (2.3.3). This integral formulation is then useful to pass limits and find bounds.

An additional interesting property of Pólya ensembles is that the corresponding probability density on the eigenvalues (of the corresponding bi-unitarily invariant ensemble) is also determinantal, see [91, Theorem 3.5 and Lemma 4.1]. In particular, for the eigenvalues the probability density is given by

$$f_{\mathrm{EV}}(z_1,\dots,z_n) = C_{\mathrm{EV}}|\Delta_n(z)|^2 \prod_{j=1}^{n} w_n\left(|z_j|^2\right), \tag{2.1.41}$$

with

$$C_{\mathrm{EV}} = \frac{1}{\pi^n n!}\frac{1}{\prod_{j=1}^{n}\mathcal{M}w_n(j)}. \tag{2.1.42}$$

It can then be expressed in the determinantal form

$$f_{\mathrm{EV}}(z_1,\dots,z_n) = \frac{1}{n!}\det\left[K_n^{\mathrm{EV}}(z_b,z_c)\right]_{b,c=1}^{n}, \tag{2.1.43}$$

with the kernel

$$K_n^{\mathrm{EV}}(z_b,z_c) = w_n\left(|z_c|^2\right)\sum_{j=0}^{n-1}\frac{(z_b\overline{z_c})^j}{\pi\mathcal{M}w_n(j+1)}. \tag{2.1.44}$$

On the level of the squared eigenradii the density is then permanental and given by

$$(r_1,\dots,r_n)\mapsto \pi^n C_{\mathrm{EV}}\,\mathrm{perm}[r_j^{k-1}]_{j,k=1}^{n}\prod_{j=1}^{n} w_n(r_j), \tag{2.1.45}$$

where the permanent is the totally symmetric version of the determinant, where the signature of the permutations are all taken to be $+1$,

$$\mathrm{perm}\left[x_{jk}\right]_{j,k=1}^{n} := \sum_{\sigma\in \mathrm{S}_n}\prod_{j=1}^{n} x_{j\sigma(j)}, \tag{2.1.46}$$

with S_n the symmetric group of permutations of n elements. In particular, the 1-point function on the eigenradii is

$$\rho_{\mathrm{EV}}(r) = w_n(r)\sum_{j=0}^{n-1}\frac{r^j}{\mathcal{M}w_n(j+1)}. \tag{2.1.47}$$

2.1.6 Orthogonal polynomial ensembles

The last subclass we would like to introduce is the class of *orthogonal polynomial ensembles* or, in short, *orthogonal ensembles* [23, 95]. Note that while this denomination is standard—originating from the theory of orthogonal polynomials—the same denomination is also used in the literature to refer to ensembles invariant under the action of the orthogonal group $\mathrm{O}(n)$; cf. [2, 14, 39, 51, 115]. This will not lead to confusion in our context as we are focusing on bi-unitary invariant ensembles.

Definition 2.1.17 (Orthogonal ensemble) *An orthogonal ensemble is a polynomial ensemble for which there exists a weight function ω such that*

$$\mathrm{Span}\{w_{k-1}\}_{k=1}^{n} = \mathrm{Span}\{x \mapsto x^{k-1}\omega(x)\}_{k=1}^{n} \tag{2.1.48}$$

and

$$x \mapsto x^{j+k}\omega(x) \in \mathrm{L}^1(\mathbb{R}_+), \quad \forall j,k \in [\![0, n-1]\!]. \tag{2.1.49}$$

The probability density function of orthogonal ensembles can therefore be written as follows

$$f(x_1,\dots,x_n) = C_{\mathrm{SV}}\Delta_n(x)\det\left[x_j^{k-1}\omega(x_j)\right]_{j,k=1}^{n} = C_{\mathrm{SV}}\Delta_n(x)^2\prod_{j=1}^{n}\omega(x_j). \tag{2.1.50}$$

As it is a polynomial ensemble, it is also determinantal and the kernel can be expressed, using the Christoffel-Darboux formula [134, p.43], as

$$K_n(x,y) = \omega(x)\sum_{j=0}^{n-1}\mathcal{P}_j(x)\mathcal{P}_j(y) = \gamma_n\omega(x)\frac{\mathcal{P}_n(x)\mathcal{P}_{n-1}(y) - \mathcal{P}_{n-1}(x)\mathcal{P}_n(y)}{x-y}, \tag{2.1.51}$$

where $\mathcal{P}_j$ are the orthonormal polynomials with respect to the weight ω on $\mathbb{R}_+$ and

$$\gamma_n = \int_0^\infty \mathcal{P}_n(x)\mathcal{P}_{n-1}(x)dx. \tag{2.1.52}$$

The orthogonal ensemble is then usually named after the kind of polynomial on which its kernel is built. The classical Laguerre, Jacobi and Cauchy-Lorentz ensembles are both, orthogonal and Pólya ensembles. Note that orthogonal ensembles are not necessarily Pólya ensembles. However, a sufficient condition to be also a Pólya ensemble is for the weight ω to satisfy the Rodrigues' formula

$$\omega(x)\mathcal{P}_j(x) = \frac{d^j}{dx^j}\left[B(x)^j\omega(x)\right], \tag{2.1.53}$$

with B a polynomial of degree at most 2 and with at least one real root; see Lemma A.3.1. This offers an explicit alternative to the particular choice of bi-orthonormal sets of function given by (2.1.38) and one can thus take $\{\mathcal{P}_j\}_{j=0}^{n-1}$ and $\{x \mapsto \omega(x)\mathcal{P}_j(x)\}_{j=0}^{n-1}$. The Pólya weight w_n and the orthogonal weight ω are then related via Rodrigues' formula (2.1.53), see subsection 3.3.4 (3.3.34).

Orthogonal ensembles are particularly interesting because many classical ensembles belong to this class—as well as Pólya [149]—but also because, in this case, all the tools of orthogonal polynomials are then available, in particular to study the large n asymptotics. One such powerful tool is the Riemann-Hilbert problem [38, 40, 41, 50], which provides a way to compute the asymptotics of the polynomials without knowing their explicit expression. While we will not use this method in this work, it is of particular importance to compute asymptotic limits of the probability

density functions when the size of the matrices goes to infinity. Limits which are surprisingly often independent of the type of matrices considered.

We present some results about the universality of those limits in the following section, which is based on [32] and [2, Chapter 6]. More details can also be found in [17].

2.2 Double scaling limit and universality

One of the main insights of random matrix theory is the fact that large random matrices often exhibits universal properties independent of the ensemble. This offers a way to classify ensembles in universality classes. Note that there are different types of universal properties. They can be divided into two kinds: local and global. We will focus here on universal properties of eigenvalues (and singulars values). The Figure 2.1 gives an illustration of such global universality for the eigenvalues.

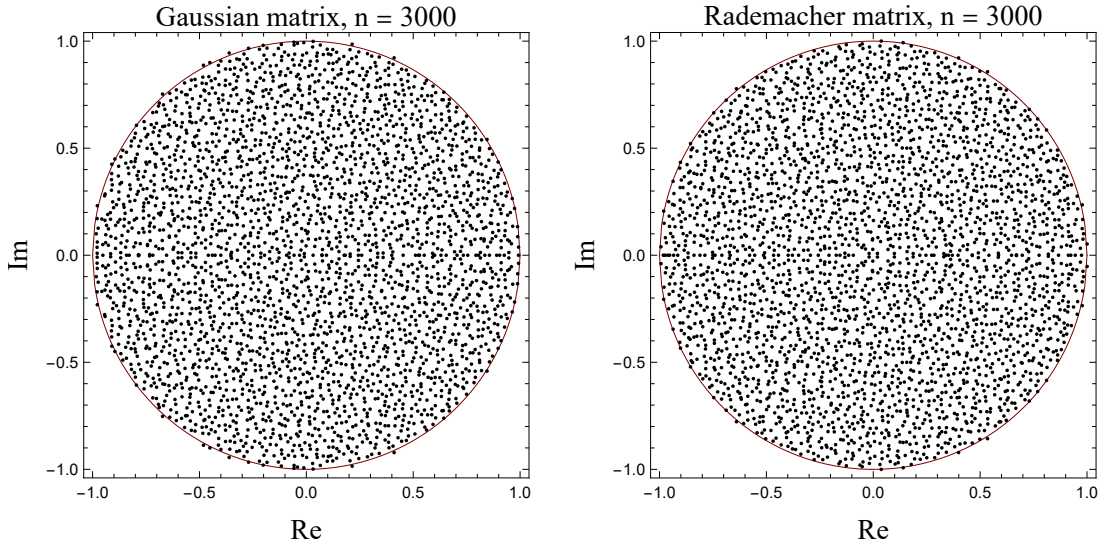


Figure 2.1: Girko's Universality Phenomenon. (Reproduced from [32])
In black, the eigenvalues of $\frac{1}{\sqrt{n}}X$, where X is a real random square matrix of size $n \times n$ with i.i.d. entries of mean 0 and variance 1 (Gaussian on the left, Rademacher on the right). In red, the unit circle in the complex plane.

To describe mathematically the distribution of the eigenvalues (and singular values) of random matrices, we introduce the *empirical distribution/measure*, also known as *non-averaged level density* in the physics literature.

Definition 2.2.1 *For a matrix $X \in \mathbb{C}^{n\times n}$ with eigenvalues $z_1, \dots, z_n \in \mathbb{C}$; which are implicit functions of X, hence the notation $z_j = z_j(X)$, the empirical distribution of the eigenvalues is the (random) probability measure on $\mathbb{C}$ defined by*

$$\wp_n(z; X) := \frac{1}{n}\sum_{j=1}^{n} \delta^{(2)}(z - z_j(X)), \tag{2.2.1}$$

where $\delta^{(2)}(z-z_j(X)) := \delta(\mathrm{Re}(z)-\mathrm{Re}(z_j(X)))\delta(\mathrm{Im}(z)-\mathrm{Im}(z_j(X)))$, with δ the Dirac delta function.

Note that this is nothing else than a normalised counting measure. Indeed, given a subset $B \subseteq \mathbb{C}$,

$$\wp_n(B; X) = \int_B d\wp_n(z; X) = \frac{\mathrm{card}\{1 \leqslant j \leqslant n \mid z_j(X) \in B\}}{n} \tag{2.2.2}$$

is the proportion of eigenvalues of the matrix X in the region B of the complex plane. Moreover, given a suitable test function ϕ,

$$\int_{\mathbb{C}} \phi(z) d\wp_n(z; X) = \frac{1}{n}\sum_{j=1}^{n} \phi\left(z_j(X)\right). \tag{2.2.3}$$

Remark 2.2.2 *Let μ be the probability measure from which X is drawn. Then, one can define the* (ensemble) averaged measure *(also referred to as 1-point (correlation) measure) as*

$$\varrho_n(z) := \int_{\mathbb{C}^{n\times n}} \wp_n(z; X) d\mu(X) \tag{2.2.4}$$

where the ensemble average is often denoted $\langle . \rangle_\mu$. When ϱ_n admits a density it is then called averaged level density *(also known as 1-point (correlation) function, see subsection 2.3.3). From there, given a measurable subset $B \subseteq \mathbb{C}$,*

$$\mathbb{P}_n(B) = \int_B d\varrho_n(z) \tag{2.2.5}$$

is the probability to find at least one eigenvalue in B, for the ensemble $(\mathbb{C}^{n\times n}, \mu)$.

Thus, while the empirical measure $\wp_n$ is associated with a unique configuration of eigenvalues (or singular values)—one sample—the averaged measure ϱ_n (1-point measure) is the average configuration over all possible samples. For small numbers of eigenvalues (or singular values) one expects their distribution to be quite dependent on the distribution of the matrix X and its entries, and their number n itself. However, as n gets large, one might wonder whether the average configuration of this point process or even the configurations themselves[3] converge towards a typical

[3]This phenomenon is called *self averaging* in the physics literature and is a manifestation of what is called *concentration of measure* in the mathematics literature; cf. [14].

configuration independent of the matrix entries and the size of the matrix.

In mathematical terms, the question is whether the sequence $\{\wp_n(z;X)\}_{n\in\mathbb{N}}$ converges in some sense to a measure $\mu^{(\infty)}$ and whether this limiting measure is shared by different ensembles. The convergence is usually understood in the weak sense of distributions i.e. for any suitable test function ϕ,

$$\lim_{n\to\infty}\int_{\mathbb{C}}\phi(z)d\wp_n(z;X)=\int_{\mathbb{C}}\phi(z)d\mu^{(\infty)}(z), \tag{2.2.6}$$

but can also happen in a stronger sense e.g. in probability or almost surely.

Remark 2.2.3 *Note that in general, one might have to rescale the spectrum for it to have compact support—or at least non-trivial bulk statistics—and consider instead the sequence $\{\wp_n(z;c_nX)\}_{n\in\mathbb{N}}$ with c_n some positive sequence. Equivalently one can consider a new matrix X' drawn from $d\mu\,(c_n^{-1}X)$ instead of $d\mu(X)$.*

The question above is relative to global (macroscopic) universality for which many strong results are known. We present some of them in the following section for which more detail can be found in [2, 14, 32, 115, 123].

2.2.1 Some results on global universality

For the following results we consider a Wigner random matrix $X=(X_{ij})_{1\leqslant i,j\leqslant n}$ whose entries X_{ij} are independent and identically distributed (i.i.d.) zero mean, complex-valued random variables, with variance $\sigma^2<\infty$.

Theorem 2.2.4 (Girko's universality and circle law) [22, 63, 138]
Almost surely, for all measurable $B\subset\mathbb{C}$, we have

$$\lim_{n\to\infty}\wp_n\left(B;\frac{1}{\sqrt{n}}X\right)=\mu^{(\infty)}(B) \tag{2.2.7}$$

where $\mu^{(\infty)}$ is the uniform distribution on the disk $\mathbb{D}_\sigma:=\{z\in\mathbb{C}\mid|z|\leqslant\sigma\}$ with density

$$z\mapsto\frac{1}{\pi\sigma^2}\mathbf{1}_{\mathbb{D}_\sigma}(z), \tag{2.2.8}$$

where $\mathbf{1}_{\mathbb{D}_\sigma}$ is the indicator function on $\mathbb{D}_\sigma$.

Theorem 2.2.5 (Wigner's universality and semi-circle law) [122, 145, 146]
Denoting $H := \frac{1}{\sqrt{2}}(X + X^\dagger)$[a], *we have, almost surely, for all measurable intervals* $I \subset \mathbb{R}$,

$$\lim_{n\to\infty} \wp_n\left(I; \frac{1}{\sqrt{n}}H\right) = \mu^{(\infty)}(I) \tag{2.2.9}$$

where $\mu^{(\infty)}$ *is the semi-circle law on* $[-2\sigma, 2\sigma]$ *with density*

$$x \mapsto \frac{\sqrt{4\sigma^2 - x^2}}{2\pi\sigma^2}\mathbf{1}_{[-2\sigma,2\sigma]}(x). \tag{2.2.10}$$

[a]The normalisation by $\sqrt{2}$ keeps the off diagonal entries of variance σ^2.

Theorem 2.2.6 (Marčenko-Pastur's universality and quarter-circle law) [17, 113]
Denoting $H := \sqrt{X^\dagger X}$, *we have, almost surely, for all measurable intervals* $I \subset \mathbb{R}_+$,

$$\lim_{n\to\infty} \wp_n\left(I; \frac{1}{\sqrt{n}}H\right) = \mu^{(\infty)}(I) \tag{2.2.11}$$

where $\mu^{(\infty)}$ *is the quarter-circle law on* $[0, 2\sigma]$ *with density*

$$x \mapsto \frac{\sqrt{4\sigma^2 - x^2}}{\pi\sigma^2}\mathbf{1}_{[0,2\sigma]}(x). \tag{2.2.12}$$

Remark 2.2.7 *Note that the eigenvalues of* $\sqrt{X^\dagger X}$ *are nothing else than the singular values of* X. *For the squared singular values,* $\mu^{(\infty)}$ *is the Marčenko-Pastur's law on* $[0, 4\sigma^2]$ *with density*

$$y \mapsto \frac{\sqrt{4\sigma^2 - y}}{2\pi\sigma^2\sqrt{y}}\mathbf{1}_{[0,4\sigma^2]}(y). \tag{2.2.13}$$

This is simply obtained from the quarter-circle law by proceeding with a change of variable $x = \sqrt{y}$.

Remark 2.2.8 *Examples of ensembles for which those theorems apply are respectively GinUE, GUE and LUE, cf. Table 2.1.*

Those three theorems are among the most salient results of random matrix theory. They remind the strong law of large numbers, in the sense that a random empirical mean converges, almost surely, to a deterministic one. However, the strong law of large numbers does not apply here as the eigenvalues are not independent. Interestingly, the limit is described, in those three cases, by a universal probability law which only depends on the variance of the matrix entries. This echoes the central limit theorem with the universal normal distribution.

What those results also show is that while the global limiting eigenvalue density is independent of the individual entries of the matrix, it is very much dependent of the ensemble. In general, even in the same invariance class (cf. Remark 2.2.8 and subsection 2.1.2) the global limiting eigenvalue (or singular value) distribution will be model-dependent, i.e. depending on the ensemble. In this sense, the global limiting eigenvalue distribution is therefore not universal.

2.2.2 Some results on local universality

Another natural question is to ask whether the local behaviour of the eigenvalues (or singular values) is universal in the large n limit. In other words, if one zooms around a point in a given region of the spectrum, is the local limiting distribution independent of the global probability distribution?

If the microscopic behaviour of the eigenvalues might be independent of the ensemble itself, one expect some dependence of the global invariance to remain. Moreover, when a microscopic limiting distribution exists, it will, in general, depends on which region of the spectrum we zoom in: a *bulk*, a *soft edge*, a *hard edge*, a *tail*, a *singular/multicritical point*. Precise definitions of those regions can be found in [2, 14, 51, 115], however, as a picture is often worth a thousand words, we give in Figure 2.2 a schematic illustration of a density in one dimension. Let us only mention that while a hard edge refers to a strict boundary, a soft edge is not strict and exhibits a light-tail on one side. Note that the limiting distribution does not necessarily have a density and can be a general measure with, for instance, some Dirac delta components.

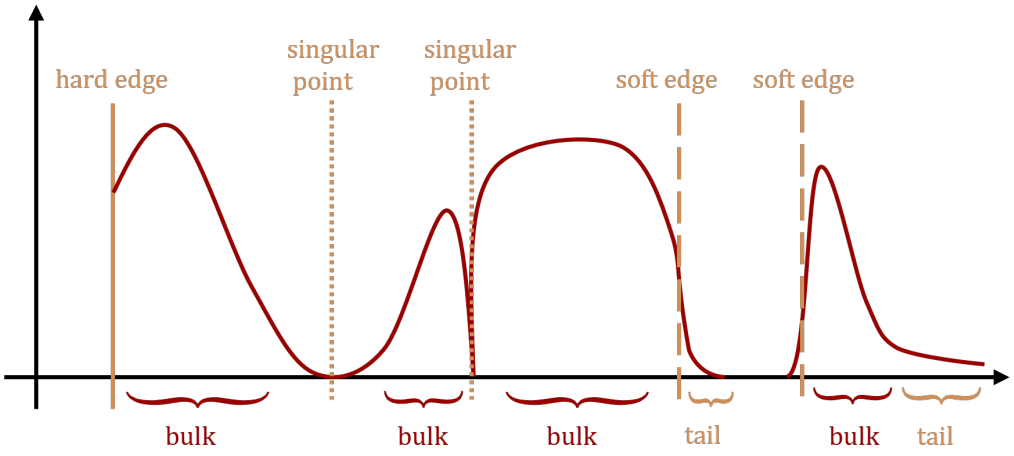


Figure 2.2: Regions of the support. Schematic representation of a probability density with one-dimensional support.

One then expects that, for ensembles which share the same symmetries—invariance of their probability distribution, the microscopic correlations between eigenvalues (singular values) in the different regions will also be shared. In this sense, the global invariance of the eigenvalue distribution determines the local correlation of the eigenvalues.

Let us focus here on unitarily invariant ensembles of the form

$$d\mu_V(H) = \frac{1}{Z_n} \exp(-n \operatorname{Tr}(V(H)))dH \tag{2.2.14}$$

defined on the space of $n \times n$ Hermitian matrices, where Z_n is the normalisation constant and dH the Lebesgue measure on this space. The potential function V is typically polynomial but can be more general. It must satisfy a growth condition to properly define a probability measure, and one can sufficiently[4] assume

$$\lim_{x \to \pm\infty} \frac{V(x)}{\ln(1+x^2)} = +\infty. \tag{2.2.15}$$

Note that the factor n in the exponential is there to balance the repulsion coming from the Vandermonde determinant (see (2.1.11) and (2.1.12)) with the confinement of the potential. This is in general enough to guarantee a non-trivial limiting global distribution of the considered spectrum; cf. Remark 2.2.3.

Because of the unitary invariance and the Jacobian of the change of coordinates (2.1.11) and (2.1.12), the underlying ensemble on the eigenvalues (and thus on the singular values) is determinantal (2.1.23). Depending on the nature of the point x_* in the spectrum under consideration, the limiting kernel of the determinantal point process

$$K_{x_*}^{(\infty)}(x,y) = \lim_{n\to\infty} \frac{1}{c_n} K_n \left(x_* + \frac{x}{c_n}, x_* + \frac{y}{c_n} \right), \tag{2.2.16}$$

when existent, may take different forms. The positive sequence $\{c_n\}_{n\in\mathbb{N}}$ is typically inversely proportional to the mean level spacing between eigenvalues (or singular values) in the considered region of the spectrum. Ensembles which share the same limiting kernel in a given region define a universality class. It is important to stress that universality classes associated to different regions of the spectrum are not necessarily the same nor disjoint. Indeed, some ensembles can share the same universality class in the bulk and belong to different universality classes at the edge—as it is often the case.

We give here examples of the limiting kernel arising when zooming in the bulk, soft edge and hard edge; more details can be found in [2, Chapter 6].

Bulk Universality. An ensemble—or more rigorously, the sequence of ensembles indexed by n—exhibits *bulk universality* at a point x_* if the limiting kernel is the *sine kernel*

$$K_{x_*}^{(\infty)}(x,y) = K_{\sin}(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)}. \tag{2.2.17}$$

This type of universality is also known as *GUE statistics*.

[4] This is sufficient but not necessary. In the case of the Cauchy-Lorentz ensemble (1.1.14), we have $\lim_{x\to\pm\infty} \frac{V(x)}{\ln(1+x^2)} = c < \infty$, $c > 0$.

Soft Edge Universality. The *soft edge universality* holds at x_* if the limiting kernel is the *Airy kernel*

$$K_{x_*}^{(\infty)}(x,y) = K^{\mathrm{Ai}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}. \tag{2.2.18}$$

Here, $x \mapsto \mathrm{Ai}(x)$ is the Airy function, which solves the Airy differential equation

$$y''(x) = xy(x), \tag{2.2.19}$$

with asymptotic behaviour, as $x \to +\infty$, given by

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}} \left(1 + o(x^{-3/2})\right). \tag{2.2.20}$$

The Airy kernel is closely related to the Tracy-Widom distribution [142], which describes the limiting distribution of the largest eigenvalue.

Hard Edge Universality. The *hard edge universality* holds at x_* if the limiting kernel is the *Bessel kernel*

$$K_{x_*}^{(\infty)}(x,y) = K_\alpha^{\mathrm{Bes}}(x,y) := \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J_\alpha'(\sqrt{x})}{2(x-y)}, \tag{2.2.21}$$

and J_α denotes the Bessel function of the first kind of order $\alpha > -1$. The parameter α governs the interaction with the hard edge: the larger α, the stronger the repulsion from the edge.

Other limiting kernels, such as the Pearcey kernel [25, 26] can arise when zooming at singular points of the spectrum. Furthermore, while we considered ensembles which were already determinantal for the macroscopic point process, the same local universal kernels can arise from ensembles which are not determinantal on the macroscopic scale and which do not even necessarily share the same global invariance of the macroscopic distribution [44–46, 64, 82, 100, 101, 106–108].

As can be seen, unitarily invariant random matrix ensembles exhibit a rich structure of universality classes. The same is true when considering the space of $n \times n$ complex matrices, instead of the Hermitian one [2, 31, 34, 35, 51, 115, 137]. In this case the eigenvalues are complex and do not coincide—up to a minus sign—anymore with the singular values. We then say that the bulk universality holds at a point $z_* \in \mathbb{C}$ if the limiting kernel is the *Ginibre kernel*

$$K_{z_*}^{(\infty)}(z,w) = K^{\mathrm{Gin}}(z,w) := \frac{1}{\pi} \exp\left(-\frac{|z|^2}{2} - \frac{|w|^2}{2} + z\overline{w}\right). \tag{2.2.22}$$

To discern broader universality classes, one may focus on quantities that are measurable and comparable in both the complex and real cases. Such quantity can be, for instance, the average distance between eigenvalues (or singular values) or the level spacings and their ratios—the ratio between de distance to the nearest

neighbour and to the next to nearest neighbour. Universality classes are thus built around these universal local gap probabilities. Remarkably, the same universal properties can arise not only within ensembles defined on the same matrix space, but also across ensembles on different matrix spaces, and even for more general operators and statistics—even in the absence of any explicit underlying operator [1, 72, 86, 129]. Note that while we presented universality results about collective behaviour of eigenvalues (singular values), some individual behaviour exist as well when looking at the largest/smallest eigenradius and singular value. Their limiting distributions are also shared between ensembles and thus define universality classes.

After this brief overview of random matrix theory, let us now present some tools from Harmonic analysis which are needed to obtain the results presented in this thesis.

2.3 Harmonic analysis

2.3.1 Notations

For the present work, we borrow most of the notations from [91]. The different matrix spaces and the corresponding measures used on them are presented in Table 2.2. First, let us recall that given a measure on $\mathrm{GL}(n,\mathbb{C})$ with a density, each of the two induced measures of the singular values and of the eigenvalues have densities, too, by Tonneli's Theorem.

Notation 2.3.1 *We will denote $f_{\mathrm{EV}} : \mathbb{C}^n \to \mathbb{R}$ the joint probability density function of the eigenvalues and $f_{\mathrm{SV}} : \mathbb{R}_+^n \to \mathbb{R}$ the joint probability density function of the squared singular values.*

The squared singular values $a = \mathrm{diag}(a_1, \dots, a_n)$ and the eigenvalues $z = \mathrm{diag}(z_1, \dots, z_n)$ will be unordered.

2.3.2 The SEV transform

Our methods rely on tools from harmonic analysis and on the bijection established in [91, Theorem 3.1] (see Theorem 2.3.5). Accordingly, we recall the relevant transforms and introduce the notation used throughout.

We start with the *Mellin transform* [125, 150], which is the multiplicative analogue of the Fourier transform and two-sided Laplace transform. The Mellin transform of a measurable function $f \in \mathrm{L}^1(\mathbb{R}_+)$, is defined as

$$\mathcal{M}f(s) := \int_0^\infty dx\; x^{s-1} f(x), \tag{2.3.1}$$

for $s \in \mathbb{C}$ such that the integral converges absolutely. When f is a probability density, the normalisation is given by $\mathcal{M}f(1) = 1$. The Mellin transform is defined at least on the line $\mathcal{C}_1 := 1 + i\mathbb{R}$. By the Mellin inversion theorem, e.g., see [91,

Matrix Space	Description	Reference Measure
$\mathrm{GL}(n,\mathbb{C})$	General linear group	$\prod_{j,k} dx_{jk}$
$[\mathrm{GL}(1,\mathbb{C})/\mathrm{U}(1)]^n \cong \mathbb{R}_+^n$	Group of positive definite diagonal matrices	$da = \prod_{k=1}^{n} da_k$
$\mathrm{GL}(1,\mathbb{C})^n \cong \mathbb{C}^n \backslash \{0\}$	Group of invertible complex diagonal matrices	$dz = \prod_{k=1}^{n} dz_k$
$\mathrm{U}(n)$	Group of unitary matrices	$d\mu_H(u)$ = normalized Haar measure
$\mathrm{T}(n)$	Group of upper unitriangular matrices	$dt = \prod_{j>k} dt_{jk}$

Table 2.2: Matrix Spaces and Reference Measures. (notation adapted from [91, Table 1]). Here, dx denotes the Lebesgue measure on $\mathbb{R}$ if x is a real variable and the Lebesgue measure $dx = d\,\mathrm{Re}\{x\} d\,\mathrm{Im}\{x\}$ on $\mathbb{C}$ if x is a complex variable.

Lemma 2.6], $\mathcal{M} : \mathrm{L}^1(\mathbb{R}_+) \to \mathcal{M}\mathrm{L}^1(\mathbb{R}_+)$ is bijective and the Mellin inversion formula can be given by the limit

$$\mathcal{M}^{-1}[\mathcal{M}f](x) := \lim_{\varepsilon \to 0} \int_{\mathcal{C}_1} \frac{ds}{2\pi i} \zeta(\varepsilon\,\mathrm{Im}\{s\}) x^{-s} \mathcal{M}f(s) = f(x), \tag{2.3.2}$$

with the regularisation ζ defined as in [91, Eq.(2.40)], where it is denoted ζ_1; in particular it is

$$\zeta(s) := \frac{\cos(s)}{1 - 4s^2/\pi^2}. \tag{2.3.3}$$

The function ζ guarantees the absolute integrability and makes the Mellin transformation bijective. It can be dropped when $\mathcal{M}f$ is absolutely integrable on $\mathcal{C}_1$. Just like the Fourier transform, the Mellin transform enjoys many properties, among which the following two are of particular interest.

Proposition 2.3.2 (Properties Mellin transform) [85, 125, 150]

- *Mellin (multiplicative) convolution. For* $f, g \in \mathrm{L}^1(\mathbb{R}_+)$,

$$\mathcal{M}(f * g)(s) = \mathcal{M}f(s)\mathcal{M}g(s), \qquad (f * g)(x) := \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}. \tag{2.3.4}$$

- *Differentiation. For an* m *times differentiable function* f *such that, for all* $k \in [\![0, m]\!]$, $x \mapsto (-x\partial_x)^k f(x) \in \mathrm{L}^1(\mathbb{R}_+)$,

$$\mathcal{M}[(-x\partial_x)^k f(x)](s) = s^k \mathcal{M}f(s), \quad k \in [\![0, m]\!]. \tag{2.3.5}$$

Notation 2.3.3 *To make the expressions more readable so that we gain more insight, we will equivalently adopt the following notation for the Mellin transform of a function* f,

$$\tilde{f}(s) := \mathcal{M}f(s) = \int_0^\infty u^{s-1} f(u) du \tag{2.3.6}$$

and its incomplete Mellin transform

$$\tilde{f}_x(s) := \int_0^x u^{s-1} f(u) du, \quad x \geqslant 0. \tag{2.3.7}$$

We also need the multivariate version of the Mellin transform, which can be defined using the tensor product $\otimes$,

$$\mathcal{M}^{\otimes n} f(s) := \int_{\mathbb{R}_+^n} da\, f(a) \prod_{k=1}^n a_k^{s_k - 1}. \tag{2.3.8}$$

As we are working with densities symmetric under permutation of their arguments, we need transformations that preserve the symmetry. Particularly, we assume $f \in \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ where $\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ is the space of symmetric Lebesgue integrable functions on $\mathbb{R}_+^n$. Those functions can be seen as (potentially signed) joint densities for the squared singular values, thus, the chosen notation. Therefore, we can go over to the symmetrised version of the multivariate Mellin transform given by

$$\mathcal{M}_{\mathrm{S}} f(s) := \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} \mathcal{M}^{\otimes n} f(\sigma(s)) = \frac{1}{n!} \int_{\mathbb{R}_+^n} da\, f(a)\, \mathrm{perm}[a_j^{s_k - 1}]_{j,k=1}^n, \tag{2.3.9}$$

with the permanent defined in (2.1.46) and S_n the finite symmetric group of permutations of n elements, $\sigma(s) = (\sigma(s_1), \ldots, \sigma(s_n))$. The symmetrized inverse Mellin transform [91, Eq.(2.39)] is, then, given by

$$\mathcal{M}_{\mathrm{S}}^{-1}[\mathcal{M}_{\mathrm{S}} f](x) = \frac{1}{n!} \lim_{\varepsilon \to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^n \frac{ds_k}{2\pi i} \zeta(\varepsilon + \mathrm{Im}\{s_k\}) \right] \mathrm{perm}[x_j^{-s_k}]_{j,k=1}^n \mathcal{M}_{\mathrm{S}} f(s), \tag{2.3.10}$$

with $\mathcal{C}(n) = \bigtimes_{k=1}^n \mathcal{C}_k$, the Cartesian product of elementary contours $\mathcal{C}_k = k + i\mathbb{R}$ which are straight lines parallel to the imaginary axis going from $k - i\infty$ to $k + i\infty$.

Another important multivariate integral transformation is the *spherical transform* $\mathcal{S} : \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+) \to \mathcal{S}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$ defined as

$$\mathcal{S}f(s) := \int_{\mathbb{R}^n_+} \left[\prod_{j=1}^n da_j\right] f(a) \frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^n}{\Delta_n(s)\Delta_n(a)} \tag{2.3.11}$$

for $s \in \mathbb{C}^n$ such that the integrand is Lebesgue integrable. We use the notation $\mathcal{S}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$ to emphasize that it is the image space of $\mathcal{S}$ with respect to the domain $\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$. Note that $\mathcal{S}$ preserves the permutation symmetry of f in its arguments.

Remark 2.3.4 *The spherical transform can be thought as the analogue of the Mellin transform for bi-unitarily invariant matrices $X \in \mathrm{GL}(n,\mathbb{C})$. To see this, let us first introduce the generalised power function*

$$(\!|X|\!)^s := \det(X)^{s_n} \prod_{j=1}^{n-1} \det(X_{j\times j})^{s_j - s_{j+1} - 1}, \quad s = (s_1, \ldots, s_n) \in \mathbb{C}^n, \tag{2.3.12}$$

where the subscript $j\times j$ denotes the top left $j\times j$ block of the matrix $X \in \mathrm{GL}(n,\mathbb{C})$. Then, making use of (2.3.12) for $x \in \mathbb{R}^n_+$, the Gelfand-Naimark integral[5] *[61] over the unitary group*

$$\int_{\mathrm{U}(n)} d\mu_H(U)\, (\!|U\,\mathrm{diag}(x)U^{-1}|\!)^s = \left(\prod_{j=0}^{n-1} j!\right) \frac{\det\left[x_j^{s_k}\right]_{j,k=1}^n}{\Delta_n(x)\Delta_n(s)} =: \chi(x;s), \tag{2.3.13}$$

enables us to rewrite the spherical transform of $f_{\mathrm{SV}} \in \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$ as a Mellin like integral on the space of positive definite Hermitian matrices $\mathrm{Herm}_+(n)$, involving the associated bi-unitarily invariant density $f_{\mathrm{BU}} \in \mathrm{L}^1(\mathrm{GL}(n,\mathbb{C}))$,

$$\begin{aligned}
\mathcal{S}f_{\mathrm{SV}}(s) &= \left(\prod_{j=0}^{n-1} \frac{1}{j!}\right) \int_{\mathbb{R}^n_+} \frac{da}{\det(a)} f_{\mathrm{SV}}(a)\chi(a;s) \\
&= \left(\prod_{j=0}^{n-1} \frac{1}{j!}\right) \int_{\mathrm{GL}(n,\mathbb{C})} \frac{dX}{|\det(X)|^{2n}} f_{\mathrm{BU}}(X)(\!|X^\dagger X|\!)^{s+n-1} \\
&= \left(\prod_{j=0}^{n-1} \frac{\pi^{j+1}}{(j!)^2}\right) \int_{\mathrm{Herm}_+(n)} \frac{dY}{\det(Y)} f_{\mathrm{BU}}\left(\sqrt{Y}\right)(\!|Y|\!)^s,
\end{aligned} \tag{2.3.14}$$

where we have used for $Y = X^\dagger X \in \mathrm{Herm}_+(n)$, $f_{\mathrm{BU}}(X) = f_{\mathrm{BU}}(\sqrt{X^\dagger X}) = f_{\mathrm{BU}}(\sqrt{Y})$. The notation $s+n-1$ is a shorthand for the vector $(s_1+n-1, \ldots, s_n+n-1)$. Note that both $Y \mapsto f_{\mathrm{BU}}(\sqrt{Y})$ and $Y \mapsto \chi(Y;s)$ are unitarily invariant (2.1.5); cf. [91, (2.14) and (2.49)].

The Gelfand-Naimark integral (2.3.13) is the multiplicative analogue of the more famous Harish-Chandra-Itzykson-Zuber (HCIZ) integral [73, 81]—which we do not

[5] If some entries of x (or s) are equal, one needs to apply l'Hôpital's rule on (2.3.13).

present here—being the analogue of the Fourier transform on the space of Hermitian matrices; more details can be found in [47, 74, 85, 140, 154].

These Mellin and spherical transforms are the building blocks for the singular value—eigenvalue transformation, coined *SEV transform* $\mathcal{R}$, defined in [91, Theorem 3.1]. It is a bijective map between the set of symmetric densities on the squared singular values, $\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$, and the set of induced densities of eigenvalues of bi-unitarily invariant matrix ensembles denoted by $\mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$. Let us stress that the bijection is valid for potentially signed densities, i.e. not necessarily probability densities. We recall the theorem here for convenience.

Theorem 2.3.5 ([91, Theorem 3.1]) *Let $\mathcal{C}(n)$ be the n-dimensional contour in (2.3.10). The map $\mathcal{R} : \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n) \to \mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$ from the joint densities of the squared singular values to the joint densities of the eigenvalues induced by the bi-unitarily invariant signed densities is bijective and has the explicit integral representation*

$$\begin{aligned} f_{\mathrm{EV}}(z) = \mathcal{R}f_{\mathrm{SV}}(z) &= \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n}|\Delta_n(z)|^2\mathcal{M}_{\mathrm{S}}^{-1}\mathcal{S}f_{\mathrm{SV}}(|z|^2) \\ &= \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n}|\Delta_n(z)|^2 \lim_{\varepsilon\to 0}\int_{\mathcal{C}(n)}\left[\prod_{k=1}^{n}\frac{ds_k}{2\pi i}\zeta(\varepsilon\,\mathrm{Im}\{s_k\})\right]\mathrm{perm}[|z_b|^{-2s_c}]_{b,c=1}^n \\ &\times \int_{\mathbb{R}_+^n}\left[\prod_{j=1}^{n}\frac{da_j}{a_j}\right] f_{\mathrm{SV}}(a)\frac{\det[a_b^{s_c}]_{b,c=1}^n}{\Delta_n(s)\Delta_n(a)}, \end{aligned} \tag{2.3.15}$$

where $|z|^2 = \mathrm{diag}(|z_1|^2, \ldots, |z_n|^2)$. Especially, $\mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n) = \mathcal{R}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$.

The explicit integral representation of the inverse map $\mathcal{R}^{-1}$ can be found in [91, Eq.(3.4)]. Let us underline that the eigenangles only appear in the factor $|\Delta_n(z)|^2$, which is independent of the ensemble. The bi-unitary invariance of the random matrix $X \in \mathrm{GL}(n,\mathbb{C})$ implies that its spectrum is isotropic which, in turn, implies that the arithmetic mean of the eigenangles should be uniformly distributed on the interval $[0, 2\pi]$. The differences of the eigenangles are, however, not uniformly distributed.

The linear integral transformations linking the different function spaces involved in Theorem 2.3.5 can be represented in the following commutative diagram which is a reduced version of [91, Eq.(3.1)],

$$\begin{array}{ccc} \mathrm{L}^{1,\mathrm{BU}}(\mathrm{GL}(n,\mathbb{C})) & \xrightarrow{\mathcal{I}_{\mathrm{SV}}} & \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n) \\ \Big\downarrow{\scriptstyle \mathcal{I}_{\mathrm{EV}}} & \swarrow{\scriptstyle \mathcal{R}} & \Big\downarrow{\scriptstyle \mathcal{S}} \\ \mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n) & \xleftarrow[\mathcal{Z}]{} & \mathcal{S}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n) \end{array} \tag{2.3.16}$$

where

$$\mathrm{L}^{1,\mathrm{BU}}(\mathbb{C}^{n\times n}) := \{\, f_{\mathrm{BU}} \in \mathrm{L}^1(\mathrm{GL}(n,\mathbb{C})) \,|\, f_{\mathrm{BU}} \text{ is bi-unitarily invariant } \}. \tag{2.3.17}$$

The transformation $\mathcal{Z} : \mathcal{S}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n) \to \mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$ is defined as

$$\mathcal{Z}\widehat{f}(z) = \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n}|\Delta_n(z)|^2\mathcal{M}_{\mathrm{S}}^{-1}\widehat{f}(|z|^2) = f_{\mathrm{EV}}(z), \tag{2.3.18}$$

while $\mathcal{I}_{\mathrm{SV}} : \mathrm{L}^{1,\mathrm{BU}}(\mathrm{GL}(n,\mathbb{C})) \to \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ consists of a singular value decomposition (1.1.1) and integrating out the unitary matrices U and V with respect to the corresponding Haar measure. Explicitly, it is

$$\mathcal{I}_{\mathrm{SV}} f_{\mathrm{BU}}(a) = \left(\frac{\pi^{n^2}}{n!\,\left(\prod_{k=0}^{n-1} k!\right)^2}\right)\Delta_n(a)^2 f_{\mathrm{BU}}(\sqrt{a}) = f_{\mathrm{SV}}(a) \tag{2.3.19}$$

and one can thus identify the bi-unitarily invariant ensemble with the corresponding ensemble of its singular values via the relation $f_{\mathrm{BU}} = \mathcal{I}_{\mathrm{SV}}^{-1} f_{\mathrm{SV}}$; cf., [92, Eq.(2.22)]. The transformation $\mathcal{I}_{\mathrm{EV}} : \mathrm{L}^{1,\mathrm{BU}}(\mathrm{GL}(n,\mathbb{C})) \to \mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$ consists of the Schur decomposition (1.1.2) and integrating out the Haar distributed unitary matrix U as well as the upper triangular matrix, i.e.,

$$\mathcal{I}_{\mathrm{EV}} f_{\mathrm{BU}}(z) = \left(\frac{1}{n!}\prod_{k=0}^{n-1}\frac{\pi^k}{k!}\right)|\Delta_n(z)|^2\left(\prod_{k=1}^{n}|z_k|^{2(n-k)}\right)\int_{\mathrm{T}(n)} dt\; f_{\mathrm{BU}}(zt) = f_{\mathrm{EV}}(z). \tag{2.3.20}$$

This latter transformation is surprisingly invertible, despite the integral over t, as shown in [91].

Remark 2.3.6 *It is worthwhile to stress that any symmetric probability density function on $\mathbb{R}_+^n$ can be traced back uniquely to a probability density of a given bi-unitarily invariant ensemble on $\mathrm{GL}(n,\mathbb{C})$. The simplest way is to build a corresponding bi-unitarily invariant matrix multiplying the matrix $a = \mathrm{diag}(a_1,\ldots,a_n)$ on the right and on the left by two independent Haar distributed unitary matrices. Unfortunately, not every symmetric probability density function on $\mathbb{C}^n$, can be seen as underlying eigenvalue distribution of bi-unitarily invariant random matrix ensemble after employing a Schur decomposition. Applying $\mathcal{R}^{-1}$ on an arbitrary symmetric probability density on $\mathbb{C}^n$ can give a signed density on $\mathbb{R}_+^n$, cf. [91, Remark 3.7]. This is why we identify ensembles on $\mathbb{R}_+^n$ —and not on $\mathbb{C}^n$—with bi-unitarily invariant ensembles on $\mathrm{GL}(n,\mathbb{C})$.*

Figure 2.3 offers a schematic representation of the different spaces of densities presented in the current section and of the subclasses of interest presented in section 2.1.

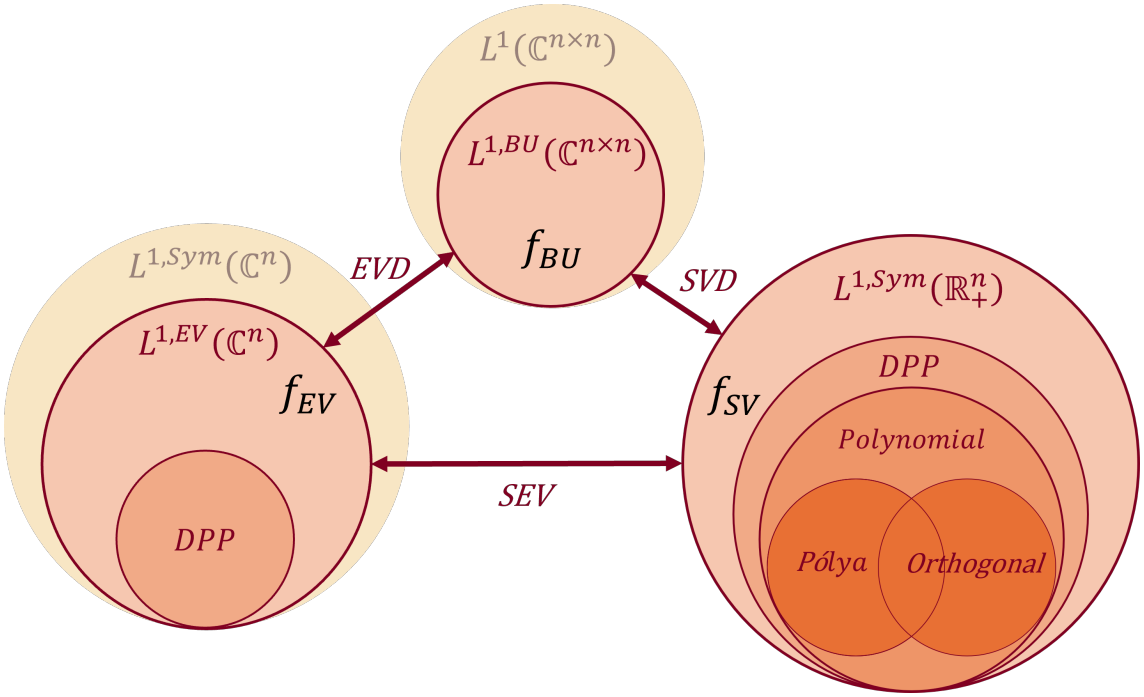


Figure 2.3: Densities and spaces. Schematic representation of the densities f_{BU} (bi-unitarily invariant), f_{SV} (squared singular values), f_{EV} (eigenvalues), and their respective spaces, along with the subclasses. DPP refers to determinantal point process. $\mathrm{L}^{1,\mathrm{Sym}}$ denotes the set of symmetric densities on the corresponding space. In particular, $\mathrm{L}^{1,\mathrm{Sym}}(\mathbb{R}^n_+) = \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$. The double arrows denote a bijective relation. The bijective maps for SEV, EVD, SVD are, respectively, the SEV transform $\mathcal{R}$, $\mathcal{I}_{\mathrm{EV}}$ and $\mathcal{I}_{\mathrm{SV}}$. The designations: polynomial, Pólya and orthogonal ensembles carry over to $\mathrm{L}^{1,\mathrm{BU}}(\mathbb{C}^{n\times n})$ under the bijection SVD; cf. Remark 2.3.6.

Our goal is to exploit Theorem 2.3.5 to explore the (collective[6]) relation between squared singular values and squared eigenradii. We would like to find the joint probability measure on both the squared eigenradii and squared singular values along with the induced marginal measures. Those are not necessarily densities as, in the case of the joint measure, equation (1.1.6) imposes a strong constraint which might give the induced measure a component of a Dirac delta measure.

2.3.3 Correlation functions

We denote the expected value of a measurable function $\phi : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$ on $\mathrm{GL}(n,\mathbb{C})$, with respect to the density f_{BU}, by

$$\mathbb{E}\left[\phi\right] := \int_{\mathrm{GL}(n,\mathbb{C})} dX\ \phi(X) f_{\mathrm{BU}}(X). \tag{2.3.21}$$

With the help of this notation we define the j,k-point correlation measures in a weak topological sense.

[6]We do not study the individual relation between specific eigenvalues and singular values but only the relation between the two full collections of those values.

Definition 2.3.7 (j,k-point correlation measure/function) [10] *Let $n \in \mathbb{N}$, $j,k \in [\![0,n]\!]$. Let f_{BU} be the probability density function of the random matrix $X \in \mathrm{GL}(n,\mathbb{C})$. Denoting the squared eigenradii of X by $\{r_l(X)\}_{l=1}^n$ and its squared singular values by $\{a_l(X)\}_{l=1}^n$, then the j,k-point correlation measure $\mu_{j,k}$ is defined weakly by the relation*

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{(n!)^2}\sum_{(\sigma,\pi)\in \mathrm{S}_n^2}\phi_{\mathrm{EV}}(r_{\sigma(1)}(X),\dots,r_{\sigma(j)}(X))\phi_{\mathrm{SV}}(a_{\pi(1)}(X),\dots,a_{\pi(k)}(X))\right]\\ &=\int_{\mathbb{R}_+^{j+k}} d\mu_{j,k}(r_1,\dots,r_j;a_1,\dots,a_k)\phi_{\mathrm{EV}}(r_1,\dots,r_j)\phi_{\mathrm{SV}}(a_1,\dots,a_k)\end{aligned} \tag{2.3.22}$$

for any Schwartz function $\phi_{\mathrm{EV}} \in \mathcal{S}\left(\mathbb{R}_+^j\right)$ and continuous bounded function $\phi_{\mathrm{SV}} \in \mathrm{C_b}\left(\mathbb{R}_+^k\right)$, where S_n is the symmetric group permuting n elements. The induced probability measures $\mu_{j,k}$, on j squared eigenradii and k squared singular values of the random matrix X are called j,k-point correlation measures. If the j,k-point correlation measure $\mu_{j,k}$ has a density with respect to the Lebesgue measure, the density will be denoted $f_{j,k}$ and will be called the j,k-point correlation function.

The measure $\mu_{j,k}$ is indeed a probability measure due to the normalization factor $1/(n!)^2$; see Remark 2.3.9. The choice of different test functions for the squared singular values and the squared eigenradii is motivated solely by technical considerations, as, for the purpose of the proofs, additional analytical properties are needed for the function on the eigenradii. We believe, however, that this distinction is of minor significance, since both bounded continuous functions and Schwartz functions are dense in all kinds of function spaces. In particular, any test function that does not factor into separate functions of the squared singular values and the squared eigenradii can be understood as a weak limit—or another appropriate type of limit—of linear combinations of product functions of the form $\phi_{\mathrm{EV}}\phi_{\mathrm{SV}}$.

Remark 2.3.8 *If $j=0$ or $k=0$, we get the marginal probability measure of only k squared singular values or j squared eigenradii, respectively. By definition we set $\mu_{0,0}=1$ so that it is consistent with $\int_{\mathbb{R}_+} d\mu_{0,1}(a) = \int_{\mathbb{R}_+} d\mu_{1,0}(r) = \mu_{0,0} = 1$.*

We would like to turn to the k-point correlation function. When one is interested in marginal densities, it is suitable to consider

$$f_k(x_1,\dots,x_k) := f_{0,k}(x_1,\dots,x_k) = \int_{\mathbb{R}_+^{n-k}} f(x_1,\dots,x_n)dx_{k+1}\dots dx_n \tag{2.3.23}$$

with $k \in [\![1,n]\!]$ and $f \in \mathrm{L}^1(A)$.

Remark 2.3.9 *This differs from the k-point correlation function R_k, used to study determinantal point processes (Definition 2.1.10), by a combinatorial factor [2, 51,*

115] reminiscent of the permutation symmetry of the density in its arguments,

$$R_k(x_1, \dots, x_k) := \frac{n!}{(n-k)!} f_k(x_1, \dots, x_k). \tag{2.3.24}$$

While f_k is a probability density, R_k is not.

We underline that the $0, k$-point and the $k, 0$-point correlation functions exist, as we are considering densities on the squared eigenradii and densities on the singular values, and those functions are k-point correlation functions (2.3.23) which can be expressed as the marginals

$$\begin{aligned} f_{j,0}(r_1, \dots, r_j) &= \int_{\mathbb{R}_+^k} f_{j,k}(r_1, \dots, r_j; a_1, \dots, a_k)\, da_1 \dots da_k, \\ f_{0,k}(a_1, \dots, a_k) &= \int_{\mathbb{R}_+^j} f_{j,k}(r_1, \dots, r_j; a_1, \dots, a_k)\, dr_1 \dots dr_k, \end{aligned} \tag{2.3.25}$$

for $j, k \in [\![1, n]\!]$. To avoid confusion we will refer to k-point correlation functions by $0, k$-point or $k, 0$-point correlation functions or state clearly whether it refers to k squared singular values or k squared eigenradii.

The probabilistic interpretation of the j, k-point correlation function is presented in Figure 2.4.

Notation 2.3.10 (Probability densities) *We will denote by*

$$\rho_{\mathrm{EV}} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad r \mapsto \rho_{\mathrm{EV}}(r) = f_{1,0}(r) \tag{2.3.26}$$

the 1-point correlation function for the squared eigenradii, and

$$\rho_{\mathrm{SV}} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad a \mapsto \rho_{\mathrm{SV}}(a) = f_{0,1}(a) \tag{2.3.27}$$

the 1-point correlation function for the squared singular values.

When studying the interaction between singular values and eigenradii, and more generally between two sets of random variables, one also wishes to know and measure how much correlated those random variables are. When the eigenvalues are independent of the singular values, the j, k-point correlation function is simply the product of the respective j-point and k-point correlation functions. Therefore, the difference between the j, k-point correlation function and this product quantifies their dependence in the general case, while excluding the internal correlations within each set. We coin this measure *cross-covariance density*.

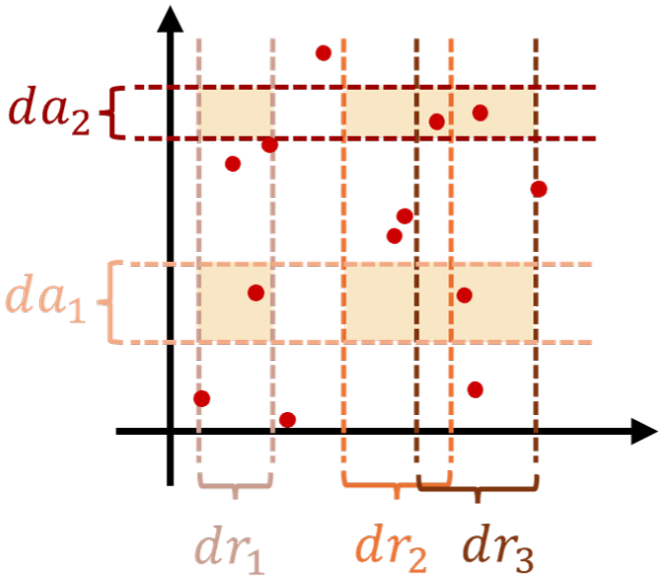


Figure 2.4: PROBABILISTIC INTERPRETATION OF j,k-POINT FUNCTIONS. The horizontal and vertical axis represents, respectively, the squared eigenradii and squared singular values coordinates, where we have represented some infinitesimal intervals of length dr_1, dr_2, dr_3 and da_1, da_2. Each particle, represented by a red dot has for coordinates a pair of squared eigenradius and squared singular value. The quantity $f_{3,2}(r_1, r_2, r_3; a_1, a_2)dr_1dr_2dr_3da_1da_2$ is thus the infinitesimal probability to find at least one particle in each shaded beige box. The same holds for any choice of $f_{j,k}$.

Definition 2.3.11 (Cross-covariance density function) [10] *The j,k-cross-covariance density function is defined as the function $\mathrm{cov}_{j,k} : \mathbb{R}_+^{j+k} \to \mathbb{R}$ given by*

$$\mathrm{cov}_{j,k}(x;y) := f_{j,k}(x;y) - f_{j,0}(x)f_{0,k}(y), \tag{2.3.28}$$

where the $f_{j,k}$, $f_{j,0}$ and $f_{0,k}$ are respectively the j,k-point, $j,0$-point and $0,k$-point correlation functions as defined in (2.3.22). For convenience, we denote $\mathrm{cov}_{1,1} := \mathrm{cov}$ and refer to it simply by cross-covariance density function.

Remark 2.3.12 *The definition of the cross-covariance is natural. Indeed, when taking a Schwartz function $\psi \in \mathcal{S}\left(\mathbb{R}_+^j\right)$ and a continuous bounded functions $\varphi \in \mathrm{C_b}\left(\mathbb{R}_+^k\right)$, we have*

$$\begin{aligned}&\int_{\mathbb{R}_+^{j+k}} (f_{j,k}(r;a) - f_{j,0}(r)f_{0,k}(a))\psi(r)\varphi(a)drda \\ =&\frac{1}{(n!)^2}\sum_{(\sigma,\pi)\in \mathrm{S}_n^2} \mathrm{Cov}\left(\psi(r_{\sigma(1)},\dots,r_{\sigma(j)}), \varphi(a_{\pi(1)},\dots,a_{\pi(k)})\right),\end{aligned} \tag{2.3.29}$$

with the covariance

$$\mathrm{Cov}(X,Y) := \mathbb{E}\Big[\Big(X - \mathbb{E}[X]\Big)\Big(Y - \mathbb{E}[Y]\Big)\Big]. \tag{2.3.30}$$

As an example, let us consider the case $j = k = 1$. When all first moments of r_j and a_k exist, it is

$$\int_{\mathbb{R}^2_+} (f_{1,1}(r;a) - f_{1,0}(r)f_{0,1}(a))ra\,dr da = \frac{1}{n^2}\sum_{j,k=1}^{n} \mathrm{Cov}(r_j, a_k), \tag{2.3.31}$$

meaning the average of the covariances between any squared eigenradius with any squared singular value. Hence, $f_{1,1} - f_{1,0}f_{0,1}$ is the average cross-covariance density between one squared singular value and one squared eigenradius.

We note that this density is very similar to the one when two variables are of the same kind. Then, the covariance density function is defined as

$$\mathrm{cov}(x_1, x_2) = f_2(x_1, x_2) - f_1(x_1)f_1(x_2), \tag{2.3.32}$$

involving the 2-point and 1-point correlation functions. In this case, the covariance density is the negative of the 2-level cluster function [42, 115], which is often used in the physics literature and also referred to as the truncated 2-point correlation function [51].

The probabilistic interpretation of the cross-covariance density is that of a covariance: it quantifies how far the two families of variables are from being statistically independent. To illustrate this, let us look at Figure 2.4 and consider, for instance, the quantity $\eta := \mathrm{cov}_{3,2}(r_1, r_2, r_3; a_1, a_2)dr_1dr_2dr_3da_1da_2$. If this quantity is positive (negative), this means it is more (less) likely to find a pair of eigenradius and singular value—a particle—in the shaded box than if the eigenradii and singular values were statistically independent. The absolute value—amplitude—of this quantity then provides a measure of the relative likelihood of the event.

An equivalent interpretation is the following. Suppose that one finds one squared singular values in each interval da_1, da_2. If $\eta > 0$ ($\eta < 0$), then it is more (less) likely one will find a squared eigenradius in each of the interval dr_1, dr_2, dr_3, compared to the case of statistical independence between eigenradii and singular values.

Thus, the regions of the support of the j,k-point function, $\mathrm{supp}(f_{j,k})$, where the j,k-cross-covariance function is positive (negative) are interpreted as attractive (repulsive) regions. The regions where the j,k-cross-covariance function vanishes indicates that the j eigenradii are close to statistical independence from the k singular values. It is, however, important to stress that, like every covariance, the cross-covariance density is weighted with the likelihood of having at all a squared singular value or a squared eigenradius in the respective intervals. The less likely it is to find one of the variables in a specific interval, the smaller the cross-covariances becomes.

To remediate this, one could naively think of introducing a density analogue of

the Pearson's correlation coefficient, of the form

$$\eta_{j,k}(x;y) := \frac{\mathrm{cov}_{j,k}(x;y)}{f_{j,0}(x)f_{0,k}(y)}. \tag{2.3.33}$$

This is indeed what is done to measure the mutual information between two random variables, as introduced by Shannon [133]. While this quantity is now standard in information theory, cf. [37], we will choose not to divide by the product of the densities to avoid technical difficulties.

We would also want to stress the fact that other quantities and functions could be used to extract the correlations between eigenradii and singular values. For instance, one could define the analogue of cluster functions/ truncated correlation functions [2, 51, 115] in the case of two different families of random variables. The generalized cluster functions would then be a particular linear combination of the j,k-point functions $f_{j,k}$.

Cluster functions are well suited to study clusters as they vanish when any (or several) of the separations between points become large compared to the mean level spacing. The n-level cluster function then captures the correlation properties of a single cluster of n levels, isolated from the more trivial contributions of lower-order correlations; cf. [42].

To start exploring the relation between eigenradii and singular values, we chose to first study the j,k-cross-covariance, which captures the correlation between the two sets of variable while excluding the internal correlations within each set. We will not study cluster functions in this work.

Chapter 3

Correlation functions between singular values and eigenvalues for fixed matrix size

A mathematician is a device for turning coffee into theorems.

Alfréd Rényi (1921–1970)

This chapter is based on a joint work with Mario Kieburg [10].

3.1 Preamble

In this chapter, we consider bi-unitarily invariant random matrices of fixed size $n \times n$ and study the collective behavior of the joint collection of eigenvalues and singular values on a global scale. More precisely, we derive exact and explicit formulas for the j,k-point measures (Definition 2.3.7) and consequently for the j,k-cross covariance measures (Definition 2.3.11). In the case of matrices of size $n = 1, 2$, we are able to get explicit formulas for all the j,k-point measures. For $n > 2$, we obtain a general formula for the $1,k$-point measures which becomes increasingly explicit as additional structural constraints are imposed on the distribution of the singular values.

The main challenges in obtaining the results presented in this chapter were first to justify the interchange of integration order between the multidimensional—both complex and real—integrals involved in the formulation of the bijective map (2.3.15) using test functions, and then to carry out most of these integrals explicitly.

3.2 Trivial cases: $n = 1, 2$

Le us start by looking at how the j,k-point correlation measures look like for the simple cases of matrices of size 1×1 (scalar) and 2×2. The following Lemma is rather helpful in relating the Definition 2.3.7 with the SEV transform (2.3.15).

Lemma 3.2.1 *Let $z(X)$ be the diagonal matrix of eigenvalues and $a(X)$ comprises the squared singular values of $X \in \mathrm{GL}(n,\mathbb{C})$. Additionally, let $f : \mathbb{C}^n \times \mathbb{R}_+^n \to \mathbb{C}$ such that $g \in \mathrm{L}^1(\mathrm{GL}(n,\mathbb{C}))$ with $g(X) = f(z(X), a(X))$. Then, we have*

$$\int_{\mathrm{GL}(n,\mathbb{C})} dX\, f(z(X),a(X)) = \frac{\pi^{n(n-1)}}{(n!)^3 \prod_{j=0}^{n-1} j!} \int_{\mathbb{C}^n} dz |\Delta_n(z)|^2 \lim_{\varepsilon\to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^{n} \frac{ds_k}{2\pi i} \zeta(\varepsilon + \mathrm{Im}\{s_k\})\right] \times \mathrm{perm}[|z_b|^{-2s_c}]_{b,c=1}^n \int_{\mathbb{R}_+^n} da\, f(z,a) \frac{\Delta_n(a) \det\left[a_b^{s_c-1}\right]_{b,c=1}^n}{\Delta_n(s)}. \tag{3.2.1}$$

We underline that f is not the joint probability density of the eigenvalues and the squared singular values but some general integrable function. It is also important to stress that the order of integration and the limit is important. Trying to exchange order of integration is what constitutes the main difficulty in getting an expression for the j,k-point functions. The rest of the difficulty being to compute explicitly those integrals.

Proof of Lemma 3.2.1. The main idea is to first decouple the integral over the first n arguments of f from the integral over the triangular matrix which appears when performing a Schur decomposition (1.1.2). Then, we can make use of the commutative diagram (2.3.16), essentially only of the triangle with the corners $\mathrm{L}^{1,\mathrm{BU}}(\mathrm{GL}(n,\mathbb{C}))$, $\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ and $\mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$. The advantage is that the complex eigenvalues z are fixed in this part of the diagram.

In the first step, we perform the Schur decomposition (1.1.2), i.e., $X = UztU^\dagger$. As the squared singular values are bi-unitarily invariant functions one can perform the integration over the unitary group and gets

$$\int_{\mathrm{GL}(n,\mathbb{C})} dX\; f(z(X),a(X)) = \int_{\mathbb{C}^n} dz \left(\frac{1}{n!}\prod_{k=0}^{n-1}\frac{\pi^k}{k!}\right) |\Delta_n(z)|^2 \left(\prod_{k=1}^{n} |z_k|^{2(n-k)}\right) \times \int_{\mathrm{T}(n)} dt f(z, a(zt)), \tag{3.2.2}$$

cf. Eq. (2.3.20). When considering the integral over t with fixed z, we notice that this is the operator $\mathcal{I}_{\mathrm{EV}}$, which is, on the other hand, equal to $\mathcal{R} \circ \mathcal{I}_{\mathrm{SV}}$, see the commutative diagram (2.3.16). We underline that the SEV transform $\mathcal{R}$ also applies for general Lebesgue integrable functions and not only probability densities due to its linear nature as an operator. As required, we assumed that $g \in \mathrm{L}^1(\mathrm{GL}(n,\mathbb{C}))$ with $g(X) = f(z(X),a(X))$ implying that f is Lebesgue integrable in the last n entries for almost all $z \in \mathbb{C}^n$ with respect to the reference measure $|\Delta_n(z)|^2 dz$. Plugging in (2.3.19) and (2.3.15) we arrive at the assertion. ■

Remark 3.2.2 *Considering Definition 2.3.7 and the joint probability density of the squared singular values $f_{\mathrm{SV}} \in \mathrm{L}^{1,\mathrm{SV}}(A)$, especially Eq. (2.3.19), we can choose, in*

particular,

$$f(z,a) = \frac{\left(\prod_{l=0}^{n-1} l!\right)^2}{\pi^{n^2} n!} \frac{f_{\mathrm{SV}}(a)}{\Delta_n(a)^2} \sum_{(\sigma,\pi)\in \mathrm{S}_n^2} \phi_{\mathrm{EV}}\left(|z_{\sigma(1)}|^2, \ldots, |z_{\sigma(j)}|^2\right) \phi_{\mathrm{SV}}\left(a_{\pi(1)}, \ldots, a_{\pi(k)}\right), \tag{3.2.3}$$

which will enable us to identify the expression of the j,k-point functions $f_{j,k}$.

In the case $j = k = 1$ with $n > 1$, it will be shown that $\mu_{1,1}$ admits a density $f_{1,1} \in \mathrm{L}^1(\mathbb{R}_+^2)$, that will therefore be called the $1,1$-point correlation function between one squared eigenradius and one squared singular value.

To illustrate the definition of j,k-point correlation measures, we consider the simplest cases of $n = 1, 2$ where we concentrate only on mixed correlation measures, meaning $j, k > 0$. For this purpose let us introduce the Dirac distribution δ, see [132], which acts on any continuous function $\phi \in \mathrm{C}^0(\mathbb{R})$ as

$$\int_{\mathbb{R}} \phi(x)\delta(x - x_0)dx = \phi(x_0) \quad \text{for almost all } x_0 \in \mathbb{R}. \tag{3.2.4}$$

3.2.1 The case: $n = 1$

Proposition 3.2.3 (The case $n = 1$) *Let $f_{\mathrm{BU}} \in \mathrm{L}^{1,\mathrm{BU}}(\mathrm{GL}(n,\mathbb{C}))$ be a probability density for $n = 1$. Then, the induced probability measure $\mu_{1,1}$ on the squared singular value and the squared eigenradius is given by*

$$d\mu_{1,1}(r_1; a_1) = \pi f_{\mathrm{BU}}(\sqrt{r_1})\delta(r_1 - a_1)dr_1 da_1. \tag{3.2.5}$$

Proof of Proposition 3.2.3. Taking $\phi \in \mathrm{C_b}\left(\mathbb{R}_+^2\right)$,

$$\mathbb{E}\Big[\phi(r_1(X); a_1(X))\Big] = \int_{\mathrm{GL}(n,\mathbb{C})} dX\ \phi(r_1(X); a_1(X)) f_{\mathrm{BU}}(X). \tag{3.2.6}$$

We proceed with a Schur decomposition (1.1.2), which is the change to polar coordinates $X = \sqrt{r_1}e^{i\theta_1}$ for $n = 1$. The measure becomes $dX = \sqrt{r_1}d\sqrt{r_1}d\theta_1 = \frac{1}{2}dr_1 d\theta_1$. We use the fact that $r_1 = a_1$ by (1.1.6), and integrate out θ_1 which is uniformly distributed by the bi-unitary invariance of X. One then gets,

$$\mathbb{E}\Big[\phi(r_1(X); a_1(X))\Big] = \pi \int_{\mathbb{R}_+} dr_1\ \phi(r_1; r_1) f_{\mathrm{BU}}(\sqrt{r_1}). \tag{3.2.7}$$

On the other hand, (2.3.22) for $\mu_{1,1}$ reads

$$\mathbb{E}\Big[\phi(r_1(X); a_1(X))\Big] = \int_{\mathbb{R}_+^2} d\mu_{1,1}(r_1; a_1)\ \phi(r_1; a_1). \tag{3.2.8}$$

Identification yields the claim. ■

3.2.2 The case: $n = 2$

The case $n = 2$ is richer, as we have now four j, k-point correlation measures with mixed statistics in the squared eigenradii and squared singular values compared to a single one for $n = 1$.

Proposition 3.2.4 (The case $n = 2$) *Let $f_{\mathrm{SV}} \in \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}^n_+)$ be the joint probability density of the squared singular values for $n = 2$. Then, the $2,2$-point correlation measure is*

$$\begin{aligned} d\mu_{2,2}(r_1, r_2; a_1, a_2) =& \Theta\left(\max\{a_1, a_2\} - \max\{r_1, r_2\}\right) \Theta\left(\min\{r_1, r_2\} - \min\{a_1, a_2\}\right) \\ &\times \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|} (r_1 + r_2)\ \delta(r_1 r_2 - a_1 a_2) dr_1 dr_2 da_1 da_2, \end{aligned} \tag{3.2.9}$$

where Θ is the Heaviside step function. The $2,1$-point correlation function is given by

$$f_{2,1}(r_1, r_2; a_1) = \left[\Theta\left(a_1 - \max\{r_1, r_2\}\right) + \Theta\left(\min\{r_1, r_2\} - a_1\right)\right] \frac{f_{\mathrm{SV}}\left(a_1, r_1 r_2 / a_1\right)}{2|a_1^2 - r_1 r_2|} (r_1 + r_2), \tag{3.2.10}$$

for almost all $r_1, r_2, a_1 > 0$ and the $1,2$-point correlation function by

$$f_{1,2}(r_1; a_1, a_2) = \Theta\left(\max\{a_1, a_2\} - r_1\right) \Theta\left(r_1 - \min\{a_1, a_2\}\right) \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|} \left(1 + \frac{a_1 a_2}{r_1^2}\right) \tag{3.2.11}$$

for almost all $r_1, a_1, a_2 > 0$. The $1,1$-point correlation function is then

$$\begin{aligned} f_{1,1}(r_1; a_1) &= \int_0^\infty da_2 f_{1,2}(r_1; a_1, a_2) = \int_0^\infty dr_2 f_{2,1}(r_1, r_2; a_1) \\ &= \left[\Theta(a_1 - r_1) \int_0^{r_1} + \Theta(r_1 - a_1) \int_{r_1}^\infty\right] da_2 \frac{f_{\mathrm{SV}}(a_1, a_2)}{2|a_1 - a_2|} \left(1 + \frac{a_1 a_2}{r_1^2}\right). \end{aligned} \tag{3.2.12}$$

Unfortunately, the marginal density $f_{1,1}$ cannot be simplified much further unless one resorts to subclasses of ensembles. For instance, polynomial ensembles have the joint probability density of the squared singular values

$$f_{\mathrm{SV}}(a_1, a_2) = \frac{(a_2 - a_1)[w_0(a_1) w_1(a_2) - w_1(a_1) w_0(a_2)]}{2[\mathcal{M}w_0(1) \mathcal{M}w_1(2) - \mathcal{M}w_0(2) \mathcal{M}w_1(1)]}, \tag{3.2.13}$$

which implies the $1,1$-point correlation function

$$\begin{aligned}
&f_{1,1}(r_1;a_1)\\
=&\frac{\Theta\left(r_1-a_1\right)}{4}\int_{r_1}^{\infty}da_2\frac{w_0(a_1)w_1(a_2)-w_1(a_1)w_0(a_2)}{\mathcal{M}w_0(1)\mathcal{M}w_1(2)-\mathcal{M}w_0(2)\mathcal{M}w_1(1)}\left(1+\frac{a_1a_2}{r_1^2}\right)\\
&-\frac{\Theta\left(a_1-r_1\right)}{4}\int_{0}^{r_1}da_2\frac{w_0(a_1)w_1(a_2)-w_1(a_1)w_0(a_2)}{\mathcal{M}w_0(1)\mathcal{M}w_1(2)-\mathcal{M}w_0(2)\mathcal{M}w_1(1)}\left(1+\frac{a_1a_2}{r_1^2}\right)\\
=&\frac{\Theta\left(r_1-a_1\right)}{4}\frac{w_0(a_1)[\mathcal{M}w_1(1)+a_1\mathcal{M}w_1(2)/r_1^2]-w_1(a_1)[\mathcal{M}w_0(1)+a_1\mathcal{M}w_0(2)/r_1^2]}{\mathcal{M}w_0(1)\mathcal{M}w_1(2)-\mathcal{M}w_0(2)\mathcal{M}w_1(1)}\\
&-\frac{1}{4}\frac{w_0(a_1)[\tilde{w}_{1,r_1}(1)+a_1\tilde{w}_{1,r_1}(2)/r_1^2]-w_1(a_1)[\tilde{w}_{0,r_1}(1)+a_1\tilde{w}_{0,r_1}(2)/r_1^2]}{\mathcal{M}w_0(1)\mathcal{M}w_1(2)-\mathcal{M}w_0(2)\mathcal{M}w_1(1)},
\end{aligned}\tag{3.2.14}$$

where $\tilde{w}_{j,r_1}$ is the incomplete Mellin transform of w_j, see Notation 2.3.3.

Proof of Proposition 3.2.4. We start from (3.2.2) with the choice (3.2.3). Without loss of generality we can assume that the test function ϕ is symmetric in its first two arguments as well as its last two ones so that the sum becomes trivial and yields a factor of 4 cancelling with the combinatorial factor in front of the sum. For $n = 2$, we use the following relation

$$a_1 + a_2 = \mathrm{tr}\big(zt(zt)^\dagger\big) = r_1(1 + |\tau|^2) + r_2, \tag{3.2.15}$$

where τ is the complex number in the off-diagonal of t. Plugging in $a_2 = r_1r_2/a_1$ originating from the identity (1.1.6), we have

$$|\tau|^2 = \frac{a_1}{r_1} + \frac{r_2}{a_1} - 1 - \frac{r_2}{r_1} \tag{3.2.16}$$

which has either two or no solutions in a_1. The situation of no solution corresponds to $a_1 \in]\min\{r_1, r_2\}, \max\{r_1, r_2\}[$ as, then, the right hand side is negative while the left hand side is non-negative. The case $a_1 \leqslant \min\{r_1, r_2\}$ corresponds to the solution with the ordering $a_1 < a_2$ while $a_1 \geqslant \max\{r_1, r_2\}$ relates to $a_1 > a_2$. Both branches map to the very same $|\tau|^2 \geqslant 0$ so that the substitution (3.2.16) is not bijective. Since the situation must be invariant under swapping a_1 and a_2 the two contributions yield the very same weight meaning it yields a factor of $1/2$.

Returning to (3.2.2), we perform a polar decomposition of τ and substitute $|\tau|$ by a_1. Afterwards, we integrate over the complex phases of τ, z_1 and z_2. Then, we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\phi(r_1(X), r_2(X); a_1(X), a_2(X))\Big]\\
=&\frac{\pi^4}{4}\int_{\mathbb{R}_+^2}dr_1dr_2(r_1+r_2)r_1\left[\int_0^{\min\{r_1,r_2\}}+\int_{\max\{r_1,r_2\}}^{\infty}\right]da_1\left|\frac{1}{r_1}-\frac{r_2}{a_1^2}\right|\\
&\times\frac{2}{\pi^4}\frac{f_{\mathrm{SV}}\left(a_1, r_1r_2/a_1\right)}{(a_1 - r_1r_2/a_1)^2}\phi\left(r_1, r_2; a_1, \frac{r_1r_2}{a_1}\right),
\end{aligned}\tag{3.2.17}$$

where we used Eq. (3.2.3) when plugging in the considered setting. From this equation and Definition 2.3.7 we can read off

$$\begin{aligned} &d\mu_{2,2}(r_1, r_2; a_1, a_2) \\ =&\left[\frac{r_1 + r_2}{a_1}\,\delta\left(\frac{r_1 r_2}{a_1} - a_2\right)\left[\Theta\left(a_1 - \max\{r_1, r_2\}\right) + \Theta\left(\min\{r_1, r_2\} - a_1\right)\right]\right. \\ &\left. + \frac{r_1 + r_2}{a_2}\,\delta\left(\frac{r_1 r_2}{a_2} - a_1\right)\left[\Theta\left(a_2 - \max\{r_1, r_2\}\right) + \Theta\left(\min\{r_1, r_2\} - a_2\right)\right]\right] \\ &\times \frac{f_{\mathrm{SV}}(a_1, a_2)}{4|a_1 - a_2|} dr_1 dr_2 da_1 da_2, \end{aligned} \tag{3.2.18}$$

where we have symmetrised in a_1 and a_2 as we consider unordered squared singular values. After applying the standard rules of the Dirac delta distribution and the Heaviside step function we find (3.2.9).

Claims (3.2.10), (3.2.11), and (3.2.12) can be readily obtained by setting, respectively,

$$\begin{aligned} \phi(r_1, r_2; a_1, a_2) &= [\psi(r_1, r_2; a_1) + \psi(r_1, r_2; a_2)]/2 \\ \phi(r_1, r_2; a_1, a_2) &= [\psi(r_1; a_1, a_2) + \psi(r_2; a_1, a_2)]/2 \end{aligned} \tag{3.2.19}$$

or

$$\phi(r_1, r_2; a_1, a_2) = [\Xi(r_1; a_1) + \Xi(r_2; a_1) + \Xi(r_1; a_2) + \Xi(r_2; a_2)]/4, \tag{3.2.20}$$

with ψ, Ξ some other test functions. ■

3.3 The $1, k$-point correlation function for $n > 2$

3.3.1 General bi-unitarily invariant ensembles

The procedure of identification used to find the joint probability measures in the previous section section 3.2 becomes very involved for $n > 2$ and has to be repeated for every different values of n. However, starting from the bijective map, given in Theorem 2.3.5, between the probability densities of the eigenvalues and singular values for bi-unitarily invariant random matrix ensembles on $\mathrm{GL}(n, \mathbb{C})$, we can derive a general expression for the $1, k$-point correlation function for $n > 2$ which is summarised in the following theorem proven in subsection 3.4.1. Unfortunately, we failed to find a general explicit formula for the j, k-point correlation function, with $j > 1$, for technical reasons discussed in section 3.5.

Theorem 3.3.1 *Given an integer $n > 2$, $k \in [\![1,n]\!]$, contours $\mathcal{C}_j = j + i\mathbb{R}$ and $(n-1)$-dimensional vectors $\tau(j) = (1,\dots,j-1,j+1,\dots,n)$ with $j \in [\![1,n]\!]$, where $[\![,]\!]$ denotes integer intervals. Let $f_{\mathrm{SV}} \in \mathrm{L}^1(\mathbb{R}_+^n)$ be the joint probability density of the squared singular values of a random matrix $X \in \mathrm{GL}(n,\mathbb{C})$ drawn from a bi-unitarily invariant ensemble. Then, the $1,k$-point correlation function $f_{1,k} : \mathbb{R}_+^{1+k} \to \mathbb{R}_+$ for a squared eigenradius r and k pairwise distinct squared singular values $a_1,\dots,a_k$ is*

$$f_{1,k}(r;a_1,\dots,a_k) = \frac{1}{n}\left(\prod_{p=0}^{n-1} p!\right)\sum_{j=1}^{n}\int_{\mathcal{C}_j}\frac{ds}{2\pi i}r^{j-1-s}\int_{\mathbb{R}_+^{n-k}}\left[\prod_{b=k+1}^{n} da_b\right] \times f_{\mathrm{SV}}(a)\frac{\det\left[a_b^{s-1}\ a_b^{\tau_c(j)-1}\right]_{\substack{b=1,\dots,n\\ c=1,\dots,n-1}}}{\Delta_n(s,\tau(j))\Delta_n(a)}, \tag{3.3.1}$$

where the determinant in the numerator should be read as follows: the first row is given by a_b^{s-1} with $b = 1,\dots,n$ as the column index and the last $n-1$ rows are $a_b^{\tau_c(l)}$ with $c = 1,\dots,n-1$ as the row index. Δ_n is the n-dimensional Vandermonde determinant (2.1.10). The n-dimensional vector $(s,\tau(j))$ has s as its first component and the components of $\tau(j)$ as its $(n-1)$ last entries.
When $k = n$, no singular value is integrated out, the only remaining integral is the complex one.
The result for degenerate $a_1,\dots,a_k$ follows from applying l'Hôpital's rule, as long as $k < n$ or that there is at least one distinct pair. For $k = n$, the limit to the fully degenerate case $\mathrm{diag}(a_1,\dots,a_n) \propto \mathbb{1}_n$ only exists in a weak sense.

In the fully degenerate case mentioned Theorem 3.3.1, the j,k-point measure $\mu_{j,k}$ does not have a density. Indeed, all singular values being equal to each others, it is also the case of the eigenradii—due to (1.1.10)—and the measure is then a weighted Dirac delta function.

The strategy to get to the $1,k$-point function is to fix k squared singular values in f_{SV}, and then use the bijection (2.3.15) to get to the eigenvalues. After integrating over all eigenangles, i.e., the angles of the complex phase of the eigenvalues, and all but one eigenradius we arrive at Theorem 3.3.1.

For the case $n = 1$, we have seen that the induced $1,1$-point measure does not have a density, cf. Proposition 3.2.3. The case $n = 2$ was given explicitly in Proposition 3.2.4, in particular (3.2.12). The proving techniques of these two results are very different than those for Theorem 3.3.1 as they are based on direct integration while for Theorem 3.3.1 one needs to take special care of the various integrations involved. Let us stress that the strategy used for $n = 1,2$ becomes intractable for $n > 2$ as one needs to treat all possible orderings of eigenradii and singular values by hands—the number of which grows as $(n!)^2$.

An immediate corollary of Theorem 3.3.1 is the conditional level density of the eigenradii conditioned under fixed squared singular values $a_1,\dots,a_n$ which can be obtained by dividing $f_{1,n}(r;a_1,\dots,a_n)$ by $f_{\mathrm{SV}}(a)$.

Corollary 3.3.2 *Under the setting of Theorem 3.3.1 (especially $n > 2$) and $a \in \mathbb{R}_+^n$ with pairwise distinct entries, the level density $\rho_{\mathrm{EV}}(.|a)$ of the squared eigenradii conditioned under fixed a is*

$$\begin{aligned}\rho_{\mathrm{EV}}(r|a) &= \frac{1}{n}\left(\prod_{p=0}^{n-1} p!\right)\sum_{j=1}^{n}\int_{\mathcal{C}_j}\frac{ds}{2\pi i} r^{j-1-s}\frac{\det\left[a_b^{s-1}\ a_b^{\tau_c(j)-1}\right]_{\substack{b=1,\dots,n\\ c=1,\dots,n-1}}}{\Delta_n(s,\tau(j))\Delta_n(a)} \\ &= \frac{1}{n}\partial_r \frac{\det\left[\begin{array}{c|c} 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) \\ \hline 1 & \binom{n-1}{c-1}\left(-\frac{a_b}{r}\right)^{c-1}\end{array}\right]_{b,c=1,\dots,n}}{\det\left[\binom{n-1}{c-1}\left(-\frac{a_b}{r}\right)^{c-1}\right]_{b,c=1,\dots,n}}.\end{aligned} \tag{3.3.2}$$

For degenerate squared singular values, with at least one distinct pair, one needs to apply l'Hôpital's rule. The limit to the fully degenerate case needs to be understood in the weak sense yielding the Dirac delta function

$$\rho_{\mathrm{EV}}(r|a_0,\dots,a_0) = \delta(r-a_0). \tag{3.3.3}$$

Remark 3.3.3 *The second line of (3.3.2) can be compared with the $n = 2$ result (3.2.11) which agree. This highlights that only the technical details of the proof are affected for this specific case but not the result. The normalisation can readily be checked, though one needs to be careful at $r = \infty$. The proper way to deal with this is to integrate over $[0, R]$ for an $R > \max_b\{a_b\}$ by expanding the first row and evaluating the Heaviside step function. Due to the derivative and the high vanishing order at $r = a_b$ the integral would be evaluated at $r = R$. After the remaining determinants are evaluated, one finds the R independent normalisation $\int_0^R \rho_{\mathrm{EV}}(r|a)dr = 1$ for any $R > \max_b\{a_b\}$ and, thence, for $R \to \infty$ too.*

Remark 3.3.4 *One may observe that only the first strict inequality in (1.1.10) is explicitly visible in the second line of (3.3.2). However, both inequalities are in fact encoded there. To make this clear, we first note that the zero entry in the upper-left corner of the determinant may be replaced by any function independent of r without changing the value of the determinant; see (A.2.5). Next, we add to the first row a copy of each of the remaining rows. Using the binomial identity, the first row then becomes*

$$\left(n,\ \left(1-\frac{a_b}{r}\right)^{n-1}(1-\Theta(r-a_b))\right)_{b=1,\dots,n}. \tag{3.3.4}$$

Using the identity

$$1-\Theta(x) = \Theta(-x), \qquad x \in \mathbb{R}, \tag{3.3.5}$$

makes the second inequality in (1.1.10) explicit. Finally, by the multilinearity of the

determinant with respect to its first row, one can make both conditions simultaneously apparent.

One result one can read off from (3.3.1) is the 1-point correlation function of the squared eigenradii ρ_{EV} in terms of f_{SV}, while the one for the squared singular values ρ_{SV} is immediate after integrating over $r \in \mathbb{R}_+$ for $k = 1$. Both are, by definition, marginal densities when integrating, respectively, over $a_1, \ldots, a_k$ or r in (3.3.1); cf. (2.3.25). To get a more compact expression for ρ_{EV}, one can make use of the Mellin transform $\mathcal{M}$, defined in (2.3.1), and the spherical transform $\mathcal{S}$, defined in (2.3.11).

Then, the 1-point function of the squared eigenradii ρ_{EV} is given by the following corollary, whose proof is a trivial identification of the terms in (3.3.1) with the definitions (2.3.1) and (2.3.11).

Corollary 3.3.5 *Consider the setting of Theorem 3.3.1 apart from $n \in \mathbb{N}$ which is not necessarily larger than 2. The 1-point correlation function of the squared eigenradii is given by*

$$\rho_{\mathrm{EV}}(r) = \frac{1}{n}\left(\prod_{p=0}^{n-1} p!\right) \sum_{j=1}^{n} r^{j-1} \mathcal{M}^{-1}\left[\mathcal{S} f_{\mathrm{SV}}(., \tau(j))\right](r). \tag{3.3.6}$$

The transformation $\mathcal{M}^{-1}$ is the inverse Mellin transform (2.3.2) on $\mathbb{R}_+$, acting on the function $s \mapsto \mathcal{S} f_{\mathrm{SV}}(s, \tau(j))$, where $(s, \tau(j))$ is the same n-dimensional vector as in Theorem 3.3.1.

For $n = 1$, (3.3.6) simplifies to $\rho_{\mathrm{EV}} = \rho_{\mathrm{SV}} = f_{\mathrm{SV}}$, because the one-dimensional spherical transform reduces to the Mellin transform, i.e., $\mathcal{S} = \mathcal{M}$ for $n = 1$. This is consistent with the fact the eigenradius is equal to the singular value, in this case, by (1.1.6).

One can see that while the formula of the $1, k$-point correlation function (3.3.1), given in Theorem 3.3.1, is exact, it is not very explicit. Indeed, one needs to carry out one complex contour integral and $n - k$ real integrals. A particular choice that enables to reduce the number of integrals to only two—independently of k and n—is when f_{SV} is the density of a polynomial ensemble (2.1.18).

3.3.2 Polynomial ensembles

When the underlying ensemble on the squared singular values is a polynomial ensemble, the Vandermonde determinant in the expression of f_{SV} simplifies with the Vandermonde determinant in the denominator on the RHS of (3.3.1). Then, the $n - k$ remaining integrals over the a's can be reduced to a single one, using a generalisation [90, Appendix C.1] of Andréief identity [15], given in the following proposition.

Proposition 3.3.6 *Let $l, m, n \in \mathbb{N}$ with $l \leqslant \min\{m, n\}$ and μ be a measure on a space S. Let $\{f_i\}_{i=1}^n$ and $\{g_j\}_{j=1}^m$ be two sets of functions such that*

$$\left| \int_S f_i(x) g_j(x)\, d\mu(x) \right| < \infty, \quad \forall (i,j) \in [\![1,n]\!] \times [\![1,m]\!], \tag{3.3.7}$$

and two matrices $X \in \mathbb{C}^{(n-l)\times n}$ and $Y \in \mathbb{C}^{(m-l)\times m}$. Then we have the following identity:

$$\begin{aligned} &\int_{S^l} \det \left[\begin{array}{cc} X_{ip} & f_i(x_k) \end{array} \right]_{\substack{i=1,\dots,n \\ p=1,\dots,n-l \\ k=1,\dots,l}} \det \left[\begin{array}{cc} Y_{jq} & g_j(x_k) \end{array} \right]_{\substack{j=1,\dots,m \\ q=1,\dots,m-l \\ k=1,\dots,l}} \prod_{k=1}^{l} d\mu(x_k) \\ &= (-1)^{(m-l)(n-l)} l! \det \left[\begin{array}{cc} \mathbf{0}_{(m-l)\times(n-l)} & Y_{jq} \\ X_{ip} & \int_S f_i(x) g_j(x)\, d\mu(x) \end{array} \right]_{\substack{i=1,\dots,n \\ p=1,\dots,n-l \\ j=1,\dots,m \\ q=1,\dots,m-l}}, \end{aligned} \tag{3.3.8}$$

where $\mathbf{0}_{(m-l)\times(n-l)}$ is the zero matrix of size $(m-l)\times(n-l)$.
The original Andréief identity is obtained by setting $l = m = n$ and reads

$$\int_{S^n} \det\left[f_i(x_j)\right]_{i,j=1}^n \det\left[g_i(x_j)\right]_{i,j=1}^n d\mu(x_1) \cdots d\mu(x_n) = n! \det\left[\int_S f_i(x) g_j(x)\, d\mu(x) \right]_{i,j=1}^n. \tag{3.3.9}$$

The general formula for the $1, k$-point function can then be expressed in the form presented in Theorem 3.3.7, which only involves the kernel K_n of the determinantal point process followed by the squared singular values and a universal—independent of the ensemble—function φ_n. While the structure of φ_n is inherited from the conditional density (3.3.2), we are not aware of any clear physical interpretation of this function—let alone a probabilistic one, since φ_n is signed. We would like to emphasize that the formulation (3.3.10) represents a tremendous improvement from (3.3.1) as it involves only two real integrals and this independently of k and n. This makes the large n asymptotic analysis possible as the number of integrations no longer grows with n. This will be the object of Chapter 4 and Chapter 5.

Theorem 3.3.7 *Let $n \in \mathbb{N}$, $n > 2$, $k \in [\![1, n]\!]$ and consider a random matrix that is drawn from a bi-unitarily invariant ensemble on $\mathrm{GL}(n, \mathbb{C})$ having a polynomial ensemble with joint probability density (2.1.23) for the squared singular values. The $1,k$-point correlation function between one squared eigenradius and one squared singular value is given by*

$$\begin{aligned} &f_{1,k}(r; a_1, \dots, a_k) \\ &= \frac{(n-k)!}{n!} \partial_r \int_0^\infty dt \int_0^r dv \frac{\left(1 - \frac{v}{r}\right)^{n-1}}{(1+t)^{n+1}} \det \left[\begin{array}{c|c} K_n(v, -rt) & K_n(v, a_c) - \delta(v - a_c) \\ \hline K_n(a_b, -rt) & K_n(a_b, a_c) \end{array} \right]_{b,c=1}^k \\ &= \frac{(n-k)!}{(n-1)!} \int_0^\infty dt \int_0^r \frac{dv}{v} \varphi_n\left(\frac{v}{r}, t\right) \det \left[\begin{array}{c|c} K_n(v, -rt) & K_n(v, a_c) - \delta(v - a_c) \\ \hline K_n(a_b, -rt) & K_n(a_b, a_c) \end{array} \right]_{b,c=1}^k, \end{aligned} \tag{3.3.10}$$

with δ the Dirac delta function,

$$\varphi_n(x,t) := x(1-x)^{n-2}(1+t)^{-(n+2)} \left[\left(1 - \frac{x}{n}\right)(1+t) - (1-x)\left(1 + \frac{1}{n}\right) \right]. \tag{3.3.11}$$

The function K_n is the the correlation kernel of the polynomial ensemble chosen to be a polynomial of degree $n-1$ in its second argument.
The 1-point function of the squared singular values is

$$\rho_{\mathrm{SV}}(a) := f_{0,1}(a) = \frac{1}{n} K_n(a, a) \tag{3.3.12}$$

and the 1-point function on the squared eigenradii is

$$\begin{aligned} \rho_{\mathrm{EV}}(r) := f_{1,0}(r) &= \partial_r \int_0^\infty \frac{dt}{(1+t)^{n+1}} \int_0^r dv \left(1 - \frac{v}{r}\right)^{n-1} K_n(v, -rt) \\ &= n \int_0^\infty dt \int_0^r \frac{dv}{v} \varphi_n\left(\frac{v}{r}, t\right) K_n(v, -rt). \end{aligned} \tag{3.3.13}$$

Remark 3.3.8 *The 1,k-point correlation function can be recast into the form*

$$f_{1,k}(r; a_1, \dots, a_k) = \rho_{\mathrm{EV}}(r) f_{0,k}(a_1, \dots, a_k) + \mathrm{cov}_{1,k}(r; a_1, \dots, a_k), \tag{3.3.14}$$

where $f_{0,k}$ is, apart from the factor $(n-k)!/n!$, the standard k-point correlation function of the squared singular values. The function $\mathrm{cov}_{1,k}$ is the $1,k$-cross-covariance density function defined in Definition 2.3.11.

For $k = 1$ Theorem 3.3.7 yields 1,1-point correlation

$$\begin{aligned} f_{1,1}(r;a) &= \int_0^\infty dt \int_0^r \frac{dv}{v} \varphi_n\left(\frac{v}{r}, t\right) \det \begin{pmatrix} K_n(v,-rt) & K_n(v,a) - \delta(v-a) \\ K_n(a,-rt) & K_n(a,a) \end{pmatrix} \\ &= \rho_{\mathrm{EV}}(r)\rho_{\mathrm{SV}}(a) + \mathrm{cov}(r;a). \end{aligned} \tag{3.3.15}$$

with

$$\begin{aligned} \mathrm{cov}(r;a) &:= \mathrm{cov}_{1,1}(r;a) \\ &= -\int_0^\infty dt\, K_n(a,-rt) \int_0^r \frac{dv}{v} \varphi_n\left(\frac{v}{r}, t\right) [K_n(v,a) - \delta(v-a)] \\ &= -\int_0^\infty dt\, K_n(a,-rt) \left[\int_0^r \frac{dv}{v} \varphi_n\left(\frac{v}{r}, t\right) K_n(v,a) - \frac{\Theta(r-a)}{a} \varphi_n\left(\frac{a}{r}, t\right)\right]. \end{aligned} \tag{3.3.16}$$

The explicit expression (3.3.13) for the 1-point function ρ_{EV} of the squared eigenradii is new for a general polynomial ensemble. It was, however, known for Pólya ensembles, as in this case, the double integral can be carried out; cf. [91, Eq.(4.7)].

3.3.3 Pólya ensembles

As just mentioned, when the underlying polynomial ensemble on the squared singular values belong to the subclass of Pólya ensembles, one can explicitly carry out the double integral in the formulation the $1,k$-point correlation function $f_{1,k}$ (3.3.10). Actually, only the $1,k$-cross-covariance function is new for this subclass of ensembles and will be stated in the following corollary, proven in subsection 3.4.4. The respective densities on the squared eigenradii and squared singular values are given in subsection 2.1.5.

Corollary 3.3.9 *Let $n \in \mathbb{N}$, $n > 2$. With the same assumptions and notations as in Theorem 3.3.7, we assume that f_{SV} is the joint probability density of the squared singular values of a Pólya ensemble associated to an n-times differentiable weight function $w_n \in \mathrm{C}^n(\mathbb{R}_+)$. Then, the $1,k$-cross-covariance density function has the form*

$$\mathrm{cov}_{1,k}(r;a_1,\ldots,a_k) = \frac{(n-k)!}{(n-1)!}\,\partial_\mu \det[K_n(a_b,a_c) + \mu\,\mathbf{C}_n(r;a_b,a_c)]_{b,c=1}^{k}\Big|_{\mu=0} \tag{3.3.17}$$

with

$$\mathbf{C}_n(r;a_1,a_2) = \sum_{\gamma=0,1} H_\gamma(r,a_2)\left[\Theta(r-a_1)\frac{1}{a_1}\Psi_\gamma\left(\frac{a_1}{r}\right) - V_\gamma(r,a_1)\right] \tag{3.3.18}$$

with Θ the Heaviside step function and for $\gamma = 0,1$ we have employed the functions

$$\Psi_\gamma(x) := \left(\frac{1-x}{nx-1}\right)^\gamma x(1-x)^{n-2}\,(nx-1)\,, \tag{3.3.19}$$

$$H_\gamma(x,y) := \int_0^1 du\; q_n(yu)\partial_u^\gamma\left[u^\gamma\frac{\rho_{\mathrm{EV}}(xu)}{w_n(xu)}\right], \tag{3.3.20}$$

$$V_\gamma(x,y) = \int_0^1 du\; p_{n-1}\,(yu)\,(u\partial_u)^{1-\gamma}\,w_n(xu), \tag{3.3.21}$$

where p_{n-1} and q_n are the bi-orthonormal pair of functions composing the kernel (2.1.35) of f_{SV} which can be expressed as

$$p_{n-1}(x) = \sum_{c=0}^{n-1}\binom{n-1}{c}\frac{(-x)^c}{\mathcal{M}w_n(c+1)} \quad \text{and} \quad q_n(x) = \frac{1}{n!}\partial_x^n[x^n w_n(x)] \tag{3.3.22}$$

according to [91, Lemma 4.2].

One can go even further and carry out the remaining integrals, $H_\gamma(x,y)$ and $V_\gamma(x,y)$, to get a computationally efficient formula in order to create plots, cf. Corollary 3.4.3. Especially, for the classical Pólya ensembles like Jacobi, Laguerre or Cauchy-Lorentz ensembles this is manageable. The formulation of Corollary 3.3.9 is useful for the asymptotic study $n \to \infty$, see Corollary 4.3.15.

When f_{SV} is the density of a Pólya ensemble, the additional structure we have from a general polynomial ensemble imposes differentiability conditions on the kernel K_n and, as a consequence, imposes continuity and differentiability conditions on the $1,k$-point correlation function $f_{1,k}$. We have analysed the analytical behaviour and proved the following conclusion in subsection 3.4.5.

Corollary 3.3.10 *Let $n \in \mathbb{N}$, $n > 1$. With the same assumptions and notations as in Theorem 3.3.7, if f_{SV} is the density of a Pólya ensemble with a weight function $w_n \in \mathrm{C}^\infty(\sigma)$ which is smooth on the support $\sigma \subset \mathbb{R}_+$ of the 1-point function ρ_{SV}, then, for $n = 2$, $f_{1,k}$ is discontinuous. For $n \geqslant 3$, it is $f_{1,k} \in \mathrm{C}^{n-3}(\sigma^{k+1})$ while it is not $(n-2)$-times continuous differentiable along $a_j = r$ for any $j = 1, \ldots, k$.*

3.3.4 Examples: Laguerre, Jacobi and Cauchy-Lorentz ensembles

Let us look at two classical examples of random matrix ensembles belonging to the Pólya class (and also to the orthogonal class). The first one is the Laguerre ensemble (1.1.12); see [91, Examples 3.4]. The associated Pólya weight function w_n is identical to the associated orthogonal weight ω and are explicitly given by

$$w_n(x) = w_{\mathrm{Lag}}(x) = x^\alpha e^{-x}, \qquad \omega(x) = \omega_{\mathrm{Lag}}(x) = x^\alpha e^{-x}, \quad \alpha > -1. \tag{3.3.23}$$

The corresponding joint probability function of the squared singular values is

$$f_{\mathrm{SV}}^{\mathrm{Lag}}(a) = \frac{1}{n!}\left(\prod_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\alpha+j+1)}\right) \Delta_n(a)^2 \prod_{j=1}^{n} a_j^\alpha e^{-a_j}, \quad \alpha > -1. \tag{3.3.24}$$

Employing the Laguerre polynomials

$$L_j^{(\alpha)}(x) := \sum_{k=0}^{j} \binom{j+\alpha}{j-k} \frac{(-x)^k}{k!}, \tag{3.3.25}$$

the particular choice (2.1.38) for the bi-orthonormal set of functions composing the kernel of the determinantal point process yields

$$p_j(x) = \frac{j!}{\Gamma(j+\alpha+1)} L_j^{(\alpha)}(x) \qquad \text{and} \qquad q_j(x) = L_j^{(\alpha)}(x) x^\alpha e^{-x}. \tag{3.3.26}$$

It is well-known [99, Corollary 5.2] that the kernel can be expressed in terms of the one-fold integral

$$K_n(x,y) = \frac{n!}{\Gamma(n+\alpha)} \int_0^1 dt\; L_{n-1}^{(\alpha)}(yt) L_n^{(\alpha)}(xt) (xt)^\alpha e^{-xt}. \tag{3.3.27}$$

What is new is the cross-covariance density of a squared singular value a with a squared eigenradius r. We make use of Corollary 3.4.3 and plug in the Mellin transform of $w_n = w_{\mathrm{Lag}}$,

$$\tilde{w}_{\mathrm{Lag}}(c+1) = \Gamma(c+\alpha+1), \tag{3.3.28}$$

its incomplete Mellin transform

$$\tilde{w}_{\mathrm{Lag},x}(c+1) = \int_0^x w_{\mathrm{Lag}}(u)u^c du = \int_0^x u^{c+\alpha}e^{-u}du = \gamma(c+\alpha+1,x), \tag{3.3.29}$$

where γ is the lower incomplete Gamma function, as well as the incomplete Mellin transform of the weight q_n which is essentially a hypergeometric function

$$\begin{aligned}\tilde{q}_{n,x}(c+1) =& \frac{1}{n!}\int_0^x du\, u^c \partial_u^n [u^{n+\alpha}e^{-u}] \\ =& \frac{x^{\alpha+c+1}}{n!}\sum_{j=0}^{\infty}\frac{\Gamma(n+\alpha+j+1)}{\Gamma(\alpha+j+1)(\alpha+c+j+1)}\frac{(-x)^j}{j!} \\ =& \frac{\Gamma(n+\alpha+1)}{n!\,(\alpha+c+1)\Gamma(\alpha+1)}x^{\alpha+c+1} \\ & \times {}_2F_2(\alpha+c+1, n+\alpha+1; \alpha+1, \alpha+c+2; -x).\end{aligned} \tag{3.3.30}$$

Unfortunately, we were unable to simplify the expression further for this case.

Another classical ensemble falling into both classes of Pólya and orthogonal ensembles is the Jacobi ensemble (1.1.13) (also known as truncated unitary ensemble). In this case, the associated Pólya and orthogonal weights are respectively given by

$$\begin{aligned} w(x) &= w_{\mathrm{Jac}}(x) = x^{\alpha}(1-x)^{\beta+n-1}\Theta(1-x), \\ \omega(x) &= \omega_{\mathrm{Jac}}(x) = x^{\alpha}(1-x)^{\beta}\Theta(1-x), \end{aligned} \tag{3.3.31}$$

for $\alpha > -1, \beta > 0$; the two weights being related via Rodrigues' formula (2.1.53). The joint probability function of the squared singular values [91, Examples 3.4] is then

$$f_{\mathrm{SV}}^{\mathrm{Jac}}(a) = \frac{1}{n!}\left(\prod_{j=0}^{n-1}\frac{\Gamma(n+\alpha+\beta+j+1)}{j!\,\Gamma(\alpha+j+1)\Gamma(n+\beta)}\right)\Delta_n(a)^2\prod_{j=1}^{n}a_j^{\alpha}(1-a_j)^{\beta}\Theta(1-a_j), \tag{3.3.32}$$

for $\alpha > -1, \beta > 0$. This time, the Jacobi polynomials are involved,

$$\begin{aligned}P_j^{(\alpha,\beta)}(x) :=& \frac{1}{2^j j!\,(1-x)^{\alpha}(1+x)^{\beta}}(-\partial_x)^j[(1-x)^{j+\alpha}(1+x)^{j+\beta}] \\ =& \frac{\Gamma(j+\alpha+1)}{j!\,\Gamma(j+\alpha+\beta+1)}\sum_{k=0}^{j}\binom{j}{k}\frac{\Gamma(j+\alpha+\beta+k+1)}{\Gamma(\alpha+k+1)}\left(\frac{x-1}{2}\right)^k,\end{aligned} \tag{3.3.33}$$

see (5.3.24) for the Rodrigues formula associated to $P_j^{(\alpha,\beta)}(1-2x)$. It can be shown that, using the choice (2.1.38), we get

$$\begin{aligned} p_j(x) =& \frac{j!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(j+\alpha+1)\Gamma(n+\beta-j)}P_j^{(\alpha,\beta+n-j)}(1-2x), \\ q_j(x) =& P_j^{(\alpha,\beta+n-j-1)}(1-2x)\,x^{\alpha}(1-x)^{\beta+n-j-1}\Theta(1-x), \end{aligned} \tag{3.3.34}$$

because of the Mellin transform of $w_n = w_{\text{Jac}}$ given by the Beta function

$$\tilde{w}_{\text{Jac}}(c+1) = \frac{\Gamma(c+\alpha+1)\Gamma(n+\beta)}{\Gamma(n+\alpha+\beta+c+1)} = \mathrm{B}(\alpha+c+1, n+\beta). \tag{3.3.35}$$

We note that the resulting Jacobi polynomials are not the standard ones when approaching this ensemble with orthogonal polynomials; see Remark 2.1.12. The reason is that we constructed those via bi-orthonormality which is certainly also allowed. The kernel, thus, takes the form

$$K_n(x,y) = \frac{n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha)\Gamma(\beta-1)} \int_0^1 dt\, P_{n-1}^{(\alpha,\beta+1)}(1-2yt) P_n^{(\alpha,\beta-1)}(1-2xt)\,(xt)^\alpha (1-xt)^{\beta-1}. \tag{3.3.36}$$

see also [93, Proposition 2.7] for $r = 1$.

The incomplete Mellin transforms needed for Corollary 3.4.3 are the incomplete Beta function

$$\begin{aligned} \tilde{w}_{\text{Jac},x}(c+1) &= \int_0^x w_{\text{Jac}}(u) u^c du = \int_0^x u^{c+\alpha}(1-u)^{\beta+n-1} du \\ &= \mathrm{B}(x, c+\alpha+1, \beta+n), \end{aligned} \tag{3.3.37}$$

and, anew, a hypergeometric function

$$\begin{aligned} \tilde{q}_{n,x}(c+1) =& \frac{1}{n!} \int_0^x du\, u^c \partial_u^n [u^{n+\alpha}(1-u)^{\beta+n-1}] \\ =& \frac{x^{\alpha+c+1}}{n!} \sum_{j=0}^\infty \binom{n+\beta-1}{j} \frac{\Gamma(n+\alpha+j+1)}{\Gamma(\alpha+j+1)(\alpha+c+j+1)} (-x)^j \\ =& \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)(\alpha+c+1)} x^{\alpha+c+1} \\ & \times {}_3F_2(\alpha+c+1, n+\alpha+1, 1-n-\beta; \alpha+1, \alpha+c+2; x). \end{aligned} \tag{3.3.38}$$

In this calculation we have used $x \in]0,1[$ as this is the support of the squared eigenradii and squared singular values.

The third classical ensemble belonging to the intersection of Pólya and orthogonal ensembles is the Cauchy-Lorentz ensemble (1.1.14) for which the associated Pólya and orthognal weights are given by

$$w_n(x) = w_{\text{CL}}(x) = \frac{x^\alpha}{(1+x)^{\beta+n-1}}, \quad \omega(x) = \omega_{\text{CL}}(x) = \frac{x^\alpha}{(1+x)^\beta}, \tag{3.3.39}$$

for $\alpha > -1$, $\beta - \alpha > 1$. The Mellin transform of the Pólya weight is then

$$\tilde{w}_{\text{CL}}(c+1) = \frac{\Gamma(c+1+\alpha)\Gamma(n-2-c+\beta-\alpha)}{\Gamma(n-1+\beta)} = \mathrm{B}(c+1+\alpha, n-2-c+\beta-\alpha), \tag{3.3.40}$$

and the joint probability function of the squared singular values reads

$$f_{\mathrm{SV}}^{\mathrm{CL}}(a) = \frac{1}{n!}\left(\prod_{j=0}^{n-1}\frac{\Gamma(n-1+\beta)}{j!\,\Gamma(c+1+\alpha)\Gamma(n-2-c+\beta-\alpha)}\right)\Delta_n(a)^2\prod_{j=1}^{n}\frac{a_j^\alpha}{(1+a_j)^\beta}, \tag{3.3.41}$$

for $\alpha > -1$, $\beta - \alpha > 1$. Note that Cauchy-Lorentz ensembles are essentially Jacobi ensembles in disguise. Indeed, let us consider the density of the Jacobi ensembles (3.3.32). Under the change of variables $a_j \mapsto a_j/(1+a_j)$, one gets

$$f_{\mathrm{SV}}^{\mathrm{Jac}}(a) \propto \left(\det\left[\left(\frac{a_j}{1+a_j}\right)^{k-1}\right]_{j,k=1}^{n}\right)^2\prod_{j=1}^{n}\frac{a_j^\alpha}{(1+a_j)^{\alpha+\beta+2}}. \tag{3.3.42}$$

By pulling out a factor $(1+a_j)^{-(n-1)}$ on each column of the determinant and proceeding with elementary operations on the columns, one gets

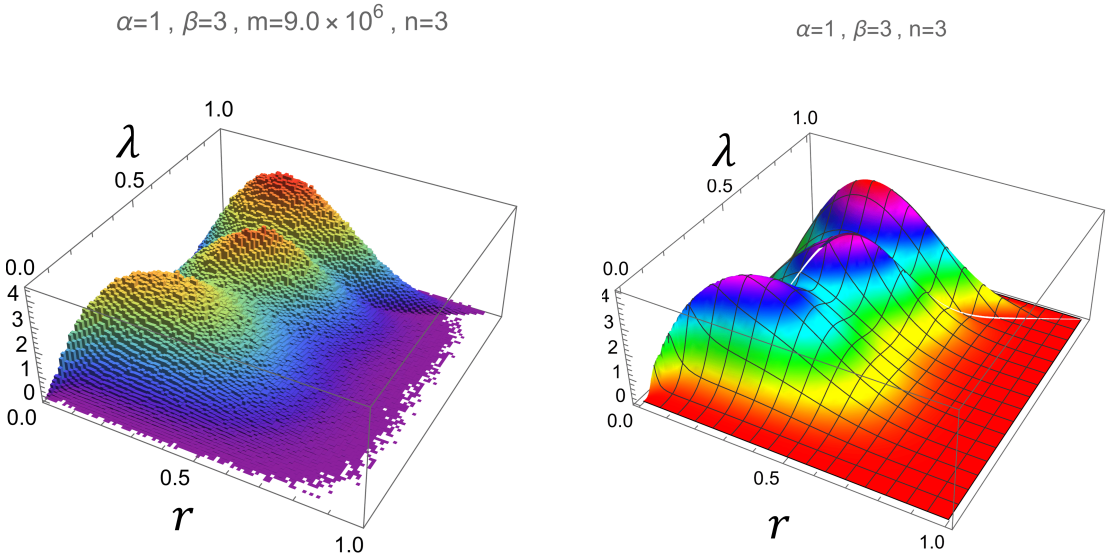
$$f_{\mathrm{SV}}^{\mathrm{Jac}}(a) \propto \Delta_n(a)^2\prod_{j=1}^{n}\frac{a_j^\alpha}{(1+a_j)^{\alpha+\beta+2n}}, \tag{3.3.43}$$

which corresponds to the density of a Cauchy-Lorentz ensemble with parameters $\alpha' = \alpha$ and $\beta' = \beta + \alpha + 2n$; the second parameter being now n-dependent. The orthonormal polynomials $Q_j^{(\alpha,\beta)}$ associated to the orthogonal weight $\omega_{\mathrm{CL}}(x) = x^\alpha/(1+x)^\beta$ are then related to Jacobi polynomials under the same change of variables and satisfy the Rodrigues formula

$$\omega_{\mathrm{CL}}(x)Q_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^n(1+x)^n\omega_{\mathrm{CL}}(x)\right]. \tag{3.3.44}$$

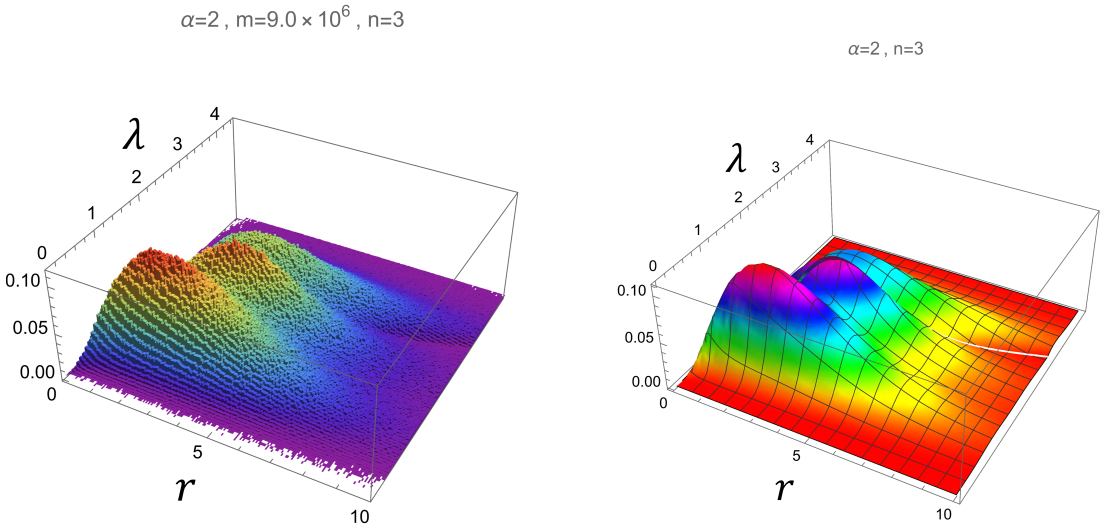
3.3.5 Plots of numerical simulations

In the case where the squared singular values of a random complex matrix follow the Jacobi or Laguerre ensemble distributions, one can use the explicit forms of the kernels given in (3.3.27) and (3.3.36) to generate numerical plots of the 1,1-point function $f_{1,1}$ and the cross-covariance density function. Furthermore, it is possible to construct random complex matrices whose squared singular values are known to follow the Jacobi or Laguerre ensembles, thereby enabling Monte-Carlo simulations that can be compared with the theoretical plots.



(a) Monte-Carlo simulation **(b)** Numerical evaluation

Figure 3.1: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Jacobi ensemble: $n = 3$, $(\alpha, \beta) = (1, 3)$. *m is the number of samples for the Monte-Carlo simulation.*



(a) Monte-Carlo simulation **(b)** Numerical evaluation

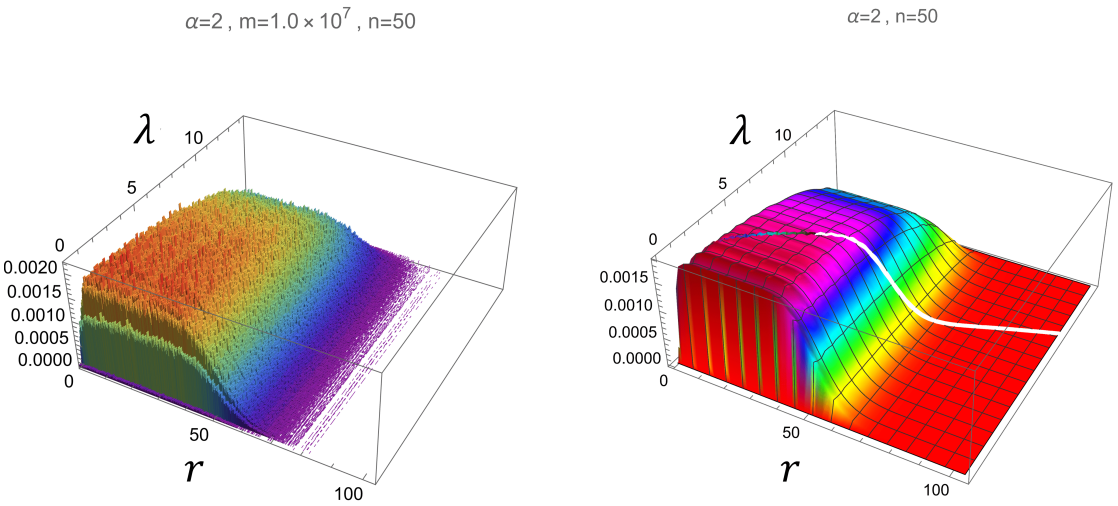
Figure 3.2: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Laguerre ensemble: $n = 3$, $\alpha = 2$. *m is the number of samples for the Monte-Carlo simulation.*

As shown in Figure 3.1 and Figure 3.2, the Monte-Carlo simulations are in complete agreement with the numerical evaluation of the formula (3.3.10).

Remark 3.3.11 *We have chosen the singular values λ instead of the squared singular values $a = \lambda^2$ as, then, the scaling in n of the mean level spacing agrees with the one of the squared eigenradii. Moreover, for the Laguerre ensemble, the limiting*

distribution of the squared singular values follows the Marčenko-Pastur's distribution (2.2.13)*, which has a square root singularity at the origin, while the singular values follow the quarter-circle law, which does not have a singularity.*

The white line visible in the numerical plots corresponds to the non-analyticity of the density function $(x, y) \mapsto f_{1,1}(x; y)$ at $x = y$. Moreover, one can see in Figure 3.1 and Figure 3.2 the presence of 3 different distinct maxima in the λ direction, this corresponds to the most likely average location of the singular values, for a given eigenradius. The maxima are distinct due to the repulsion between the singular values, which is translated mathematically by the fact the point process is determinantal (2.1.23). For the eigenradii, in the r direction, there is no distinct maximum as the process is then permanental (2.1.45); there is no repulsion between eigenradii.



(a) Monte-Carlo simulation **(b)** Numerical evaluation

Figure 3.3: Plots of $(r, \lambda) \mapsto 2\lambda f_{1,1}(r; \lambda^2)$ for the Laguerre ensemble: $n = 50$, $\alpha = 2$. *m is the number of samples for the Monte-Carlo simulation.*

For $n = 50$, Figure 3.3 shows that singular values and eigenradii becomes statistically independent on the macroscopic scale—this was already apparent in Figure 3.1 and Figure 3.2. One then retrieves the quarter-circle law distribution (2.2.12) for the singular values and the uniform distribution on $[0, n]$ for the squared eigenradii (2.2.8). While there is an apparent decoupling between the singular values and the squared eigenradii on the macroscopic scale, some correlation remains on a microscopic scale due to Weyl's inequalities (1.1.8) (1.1.9) (1.1.10), which are still strict. Hence the need to subtract the product of the respective 1-point functions to the 1, 1-point function to access these overshadowed correlations; cf. Definition 2.3.11.

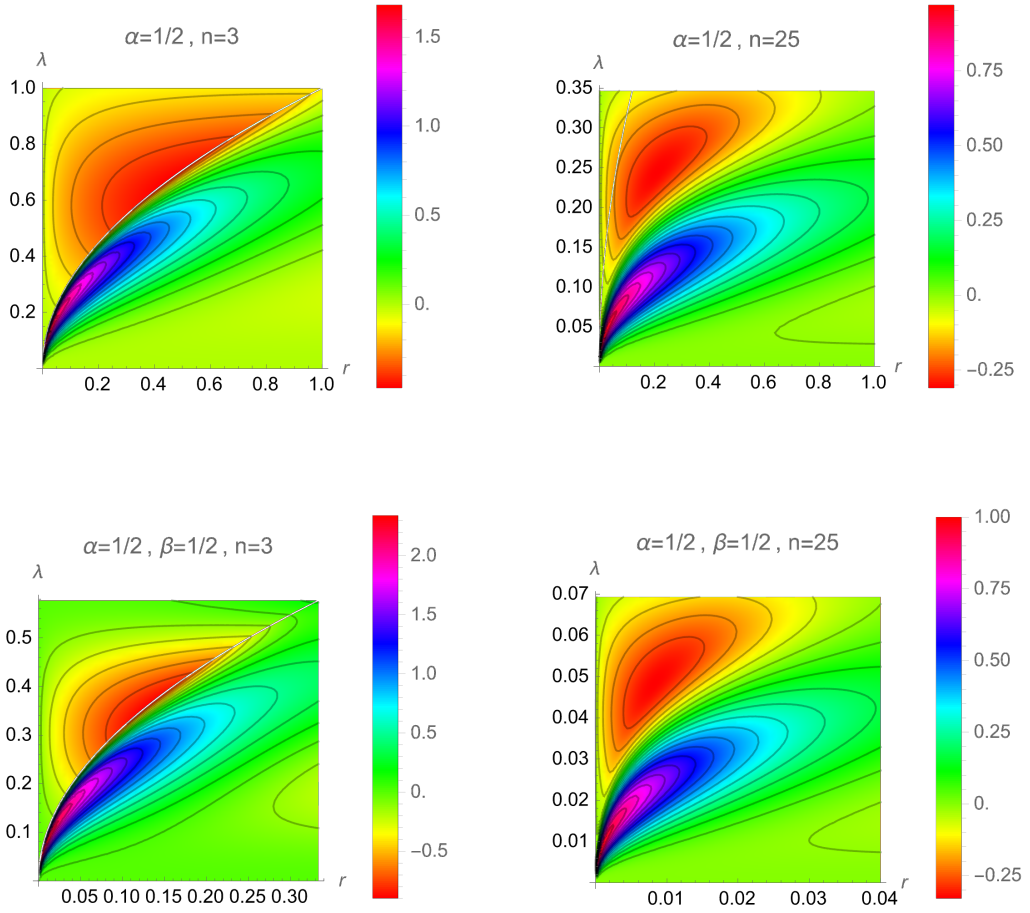


Figure 3.4: Cross-covariance at the origin. Contour plots of the cross-covariance density $(r, \lambda) \mapsto 2\lambda \mathrm{cov}(r; \lambda^2)$ for the Laguerre ensemble with $\alpha = 1/2$ (upper row) and for the Jacobi ensemble with $\alpha = \beta = 1/2$ for the matrix dimensions $n = 3$ (left column) and $n = 25$ (right column). For the Laguerre ensemble we rescaled by $n^{3/2}$. See Eqs. (2.1.33) for the corresponding weight functions and their parameter dependence in α and β, respectively. We zoomed into the scale of the local mean level spacing for the singular values λ and squared eigenradii r. The impact of the line $\lambda = \sqrt{r}$, where the cross-covariance density is not smooth, is still visible for $n = 3$ but becomes hard to detect for larger matrix size.

One expects the correlations between singular values and eigenradii to remain palpable, in the large n limit, around the origin as it is a hard edge both for singular values and eigenradii. Indeed, this can be seen in Figure 3.4 , where we show contour plots of the cross-covariance density $(r, \lambda) \mapsto 2\lambda \mathrm{cov}(r; \lambda^2)$ for a Laguerre and a Jacobi ensemble with the same parameter α, see Eqs. (2.1.33), for which we know they share the same hard edge statistics of the singular values.

The probabilistic interpretation of the cross-covariance density (cf. § below Remark 2.3.12), shown in Figure 3.4, is as follows. Negative regions indicate a form

of repulsion between the eigenradius and the singular value. Within such regions, the probability of observing a correlated variable pair is lower than it would be if the variables were independent. Conversely, positive regions indicate an effective attraction between the eigenradius and the singular value. Regions where the cross-covariance vanishes are neutral, meaning the variables are essentially uncorrelated.

Remark 3.3.12 *We emphasize that the* $1, k$*-point functions are averaged densities of the unordered eigenradii and singular values; see Remark 2.2.2. They do not capture the behavior of individual singular values or eigenradii (see Figure 2.4). Thus, if the cross-covariance vanishes in a region, this means either that no eigenradius–singular-value pair typically appears there, or that only* $o(n)$ *such pairs are correlated within that region.*

Note that there is no white regions in the plots, despite Weyl's inequalities (1.1.8), (1.1.9) and (1.1.10). The first reason being that the j, k-point correlation measures are averaged measures; cf. Remark 2.2.2. The second reason being that the eigenradii and singular values are not ordered here. Indeed, the j, k-correlation measures are fully symmetric in the eigenradii and singular values (cf. Definition 2.3.7), i.e. further averaged on all possible orderings. Moreover, we zoom in a subregion of the support of both singular values and eigenradii.

The contour plots for the matrix size $n = 25$ show two things: Firstly, non-trivial (non-factorising) spectral statistics seem to emerge in the hard edge limit. Secondly, these statistics are shared by different ensembles as they match for the two ensembles, apart from a scaling. We investigate this in the next chapter Chapter 4.

3.4 Proofs

In the ensuing proofs we make use of the following lemma to compute the n-fold integral over the eigenangles; cf. [96, Lemma 1.4].

Lemma 3.4.1 *Let* $n \in \mathbb{N}$, $n \geqslant 1$. $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$, $\theta = (\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n$.

$$\int_{[0,2\pi]^n} d\theta \; |\Delta_n(\sqrt{r}e^{i\theta})|^2 = (2\pi)^n \operatorname{perm}[r_j^{k-1}]_{j,k=1}^n \tag{3.4.1}$$

Proof. At the heart of the proof lies the integral $\int_0^{2\pi} d\vartheta \; e^{i\vartheta(j-k)} = 2\pi\delta_{j,k}$ for all $j, k \in \mathbb{Z}$. Expanding both Vandermonde determinants via the Leibniz formula, we

find the claim

$$\int_{[0,2\pi]^n} |\Delta_n(\sqrt{r}e^{i\theta})|^2 d\theta = \int_{[0,2\pi]^n} d\theta \sum_{\sigma_1,\sigma_2\in \mathrm{S}_n} \mathrm{sgn}(\sigma_1\sigma_2) \prod_{j=1}^n r_j^{(\sigma_1(j)+\sigma_2(j)-2)/2} e^{i\theta_j(\sigma_1(j)-\sigma_2(j))}$$
$$= (2\pi)^n \sum_{\sigma_1\in \mathrm{S}_n} \prod_{j=1}^n r_j^{\sigma_1(j)-1} = (2\pi)^n \,\mathrm{perm}[r_j^{k-1}]_{j,k=1}^n, \tag{3.4.2}$$

where S_n is the symmetric group permuting n elements and sgn is the signum function which is $+1$ for an even permutation and -1 for an odd one. ■

3.4.1 Proof of Theorem 3.3.1

We will assume $n > 2$ in the present section as some steps require this condition to be eligible. Moreover, we assume $f_{\mathrm{SV}} \in \mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ is a probability density function.

The most important statement of Theorem 3.3.1 is that $f_{1,k}$ is indeed a function and not only a measure. For this purpose we choose a Schwartz function $\phi_{\mathrm{EV}} \in \mathcal{S}(\mathbb{R}_+)$ and a continuous bounded function $\phi_{\mathrm{SV}} \in \mathrm{C}_b(\mathbb{R}_+^k)$ and make use of (3.2.1) in combination with the choice (3.2.3) for $j = 1$, to get

$$\mathbb{E}\left[\frac{1}{n!\,n} \sum_{\substack{\sigma\in \mathrm{S}_n \\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\right]$$
$$= \frac{\left(\prod_{j=0}^{n-1} j!\right)}{\pi^n n (n!)^3} \int_Z dz |\Delta_n(z)|^2 \lim_{\varepsilon\to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^n \frac{ds_k}{2\pi i} \zeta(\varepsilon \,\mathrm{Im}\{s_k\})\right] \mathrm{Perm}[|z_b|^{-2s_c}]_{b,c=1}^n$$
$$\times \int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a) \sum_{\substack{\sigma\in \mathrm{S}_n \\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(|z_l|^2)\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)}) \frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^n}{\Delta_n(a)\Delta_n(s)}. \tag{3.4.3}$$

We then integrate over the eigenangles using Lemma 3.4.1. Additionally, we can use the permutation symmetry in r_j to replace the sum over $\phi_{\mathrm{EV}}(r_l)\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})$ by $n\phi_{\mathrm{EV}}(r_1)\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})$. We are left with

$$\mathbb{E}\left[\frac{1}{n!\,n} \sum_{\substack{\sigma\in \mathrm{S}_n \\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\right]$$
$$= \frac{\left(\prod_{j=0}^{n-1} j!\right)}{(n!)^3} \int_{\mathbb{R}_+^n} dr\, \mathrm{Perm}[r_b^{c-1}]_{b,c=1}^n \lim_{\varepsilon\to 0} \int_{\mathcal{C}(n)} \left[\prod_{k=1}^n \frac{ds_k}{2\pi i} \zeta(\varepsilon \,\mathrm{Im}\{s_k\})\right]$$
$$\times \mathrm{Perm}[r_b^{-s_c}]_{b,c=1}^n \int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a) \sum_{\sigma\in \mathrm{S}_n} \phi_{\mathrm{EV}}(r_1)\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)}) \frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^n}{\Delta_n(a)\Delta_n(s)}. \tag{3.4.4}$$

We expand the first permanent in the first column and integrate first over $r_2,\dots,r_n$ and then over r_1. Particularly the permutation invariance of the remaining integrand

in $r_2, \dots, r_n$ tells us that each integral has $(n-1)!$ identical contributions so that

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{n!\,n}\sum_{\substack{\sigma\in \mathrm{S}_n\\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Big]\\
&= \frac{\big(\prod_{j=0}^{n-1} j!\big)}{(n!)^2 n}\sum_{l=1}^{n}\int_0^\infty dr_1\, r_1^{l-1}\Big[\prod_{j=2}^{n} dr_j r_j^{\tau_j(l)-1}\Big]\lim_{\varepsilon\to 0}\int_{\mathcal{C}(n)}\Big[\prod_{k=1}^{n}\frac{ds_k}{2\pi i}\zeta(\varepsilon\,\mathrm{Im}\{s_k\})\Big]\\
&\quad\times \mathrm{Perm}[r_b^{-s_c}]_{b,c=1}^n \int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a)\sum_{\sigma\in\mathrm{S}_n}\phi_{\mathrm{EV}}(r_1)\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})\frac{\det\big[a_b^{s_c-1}\big]_{b,c=1}^n}{\Delta_n(a)\Delta_n(s)}
\end{aligned} \tag{3.4.5}$$

with the $(n-1)$-dimensional vector $\tau(l) = (1,\dots,l-1,l+1,\dots,n)$.

We define the function

$$g(a) := \frac{1}{n!\,n} f_{\mathrm{SV}}(a)\sum_{\sigma\in\mathrm{S}_n}\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)}) \tag{3.4.6}$$

which is in $\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$. Now, we can identify the integral over a with the spherical transform (2.3.11) of g, the integral over s and the limit $\varepsilon\to 0$ with the multivariate inverse Mellin transform (2.3.10) and the integral over $r_2,\dots,r_n$ as the Mellin transform $\mathrm{id}\otimes\mathcal{M}^{n-1}$ in the last $n-1$ entries, i.e.,

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{n!\,n}\sum_{\substack{\sigma\in \mathrm{S}_n\\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Big]\\
&= \Big(\prod_{j=0}^{n-1} j!\Big)\sum_{l=1}^{n}\int_0^\infty dr_1\, r_1^{l-1}\phi_{\mathrm{EV}}(r_1)(\mathrm{id}\otimes\mathcal{M}^{n-1})\mathcal{M}_S^{-1}\mathcal{S}g(r_1,\tau(l))\\
&= \Big(\prod_{j=0}^{n-1} j!\Big)\sum_{l=1}^{n}\int_0^\infty dr_1\, r_1^{l-1}\phi_{\mathrm{EV}}(r_1)(\mathcal{M}^{-1}\otimes\mathrm{id}^{\otimes n-1})\mathcal{S}g(r_1,\tau(l)).
\end{aligned} \tag{3.4.7}$$

Note that $r_1\mapsto r_1^{l-1}(\mathcal{M}^{-1}\otimes\mathrm{id}^{\otimes n-1})\mathcal{S}g(r_1,\tau(l))\in\mathrm{L}^1(\mathbb{R}_+)$ due to (2.3.16), as $g\in\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$. When writing the RHS of (3.4.7) explicitly we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{n!\,n}\sum_{\substack{\sigma\in \mathrm{S}_n\\ 1\leqslant l\leqslant n}} \phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Big]\\
&= \frac{\big(\prod_{j=0}^{n-1} j!\big)}{n!\,n}\sum_{l=1}^{n}\int_0^\infty dr_1\, r_1^{l-1}\phi_{\mathrm{EV}}(r_1)\lim_{\varepsilon\to 0}\int_{\mathcal{C}_l}\frac{ds_l}{2i\pi}\zeta(\varepsilon\,\mathrm{Im}\{s_l\})r_1^{-s_l}\\
&\quad\times\int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a)\sum_{\sigma\in\mathrm{S}_n}\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})\frac{\det\Big[a_b^{s_l-1},\ a_b^{\tau_c(l)-1}\Big]_{\substack{b=1,\dots,n\\ c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-s_l)},
\end{aligned} \tag{3.4.8}$$

where we recall that $\mathcal{C}_l = l + i\mathbb{R}$ and where we have used the fact

$$\Delta_n(s_l, \tau(l)) = \Delta_{n-1}(\tau(l)) \prod_{j=1}^{n-1} (\tau_j(l) - s_l). \tag{3.4.9}$$

To carry out the limit $\varepsilon \to 0$ ones needs to be aware that the integral over s_l is not necessarily absolutely convergent, namely in the case of degenerate a. We will see that the following order of integration

$$\lim_{\varepsilon \to 0} \int da \int dr_1 \int ds_l. \tag{3.4.10}$$

enables us to carry out the limit explicitly with the help of Lebesgue's dominated convergence theorem. To achieve this order, we first need to show that the limit $\varepsilon \to 0$ can be interchanged with the integration over r_1. The Schwartz function ϕ_{EV} and the regularisation function ζ will help us achieve it. The idea is to rewrite the complex integral on the RHS of (3.4.8) in the form of a multiplicative convolution. Indeed, the complex integral over s_l itself (without the limit $\varepsilon \to 0$) corresponds to the usual (unregularized) inverse Mellin transform of the product of the two functions

$$\tilde{\zeta}_\varepsilon(s) := \zeta(\varepsilon \operatorname{Im}\{s\}), \quad \tilde{g}_l(s) := \mathcal{S}g(s, \tau(l)). \tag{3.4.11}$$

For the sake of clarity, we will denote by $\mathcal{M}_*^{-1}$ the unregularised inverse Mellin transform. The integrals on the RHS of (3.4.8) can then be written in the compact form

$$\mathcal{M}\left[\phi_{\text{EV}} \cdot \lim_{\varepsilon \to 0} \mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon \cdot \tilde{g}_l]\right](l). \tag{3.4.12}$$

Using the Mellin inversion theorem, the function $\mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon \cdot \tilde{g}_l]$ can expressed in terms of a Mellin convolution

$$\int_{\mathcal{C}_l} \frac{ds_l}{2i\pi} r_1^{-s_l} \tilde{\zeta}_\varepsilon(s_l) \tilde{g}_l(s_l) = \mathcal{M}^{-1}[\tilde{\zeta}_\varepsilon \cdot \tilde{g}_l](r_1) = \int_0^\infty \frac{dy}{y} \mathcal{M}_*^{-1} \tilde{\zeta}_\varepsilon(y) \mathcal{M}^{-1} \tilde{g}_l \left(\frac{r_1}{y}\right), \tag{3.4.13}$$

where the inverse Mellin transform of $\tilde{g}_l$ remains regularised and implicit, i.e. $\mathcal{M}^{-1}\tilde{g}_l(y) = (\mathcal{M}^{-1} \otimes \mathrm{id}^{\otimes n-1})\mathcal{S}g(y, \tau(l))$, and the inverse Mellin transform of the auxiliary function $\tilde{\zeta}_\varepsilon$, see (2.3.3)), is

$$\begin{aligned} \mathcal{M}_*^{-1} \tilde{\zeta}_\varepsilon(x) &= \int_{\mathcal{C}_l} \frac{ds_l}{2i\pi} \zeta(\varepsilon \operatorname{Im}\{s_l\}) x^{-s_l} \\ &= \frac{\pi}{4\varepsilon} x^{-l} \cos\left(\frac{\pi \ln(x)}{2\varepsilon}\right) \left(\Theta\left(e^{\varepsilon} - x\right) - \Theta\left(e^{-\varepsilon} - x\right)\right), \end{aligned} \tag{3.4.14}$$

with Θ the Heaviside step function. We then have implicitly added another regularisation through $\mathcal{M}^{-1}\tilde{g}_l$.

Thus, combining (3.4.13) and (3.4.14), and applying already the Heaviside step

functions, we have

$$\begin{aligned}&\mathcal{M}\left[\phi_{\mathrm{EV}}\cdot\mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon\cdot\tilde{g}_l]\right](l)\\&=\int_0^\infty dr_1\, r_1^{l-1}\phi_{\mathrm{EV}}(r_1)\int_{e^{-\varepsilon}}^{e^{\varepsilon}}\frac{dy}{y}\frac{\pi}{4\varepsilon}y^{-l}\cos\left(\frac{\pi\ln(y)}{2\varepsilon}\right)\mathcal{M}^{-1}\tilde{g}_l\left(\frac{r_1}{y}\right).\end{aligned}\tag{3.4.15}$$

Then, one can interchange the two integrals. This is allowed due to the integrability of ϕ_{EV} and that $r_1\mapsto r_1^{l-1}\mathcal{M}^{-1}\tilde{g}_l(r_1)\in\mathrm{L}^1(\mathbb{R}_+)$, as mentioned earlier. The remaining factor is independent of r_1 and bounded on $[e^{-\varepsilon},e^{\varepsilon}]$. Performing the consecutive changes of variable $r_1\mapsto r_1y$ and $y=e^{\varepsilon x}$ yields

$$\begin{aligned}&\mathcal{M}\left[\phi_{\mathrm{EV}}\cdot\mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon\cdot\tilde{g}_l]\right](l)\\&=\int_0^\infty dr_1\, r_1^{l-1}\left[\int_{-1}^{1}dx\,\phi_{\mathrm{EV}}(e^{\varepsilon x}r_1)\frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right)\right](\mathcal{M}^{-1}\otimes\mathrm{id}^{\otimes n-1})\mathcal{S}g(r_1,\tau(l)),\end{aligned}\tag{3.4.16}$$

which can be compared to the integrals (3.4.8), where the only difference is the absence of $\lim_{\varepsilon\to0}$. Taking now the limit $\varepsilon\to0$ on the outside of (3.4.16), it is now clear that one can use Lebesgue's dominated convergence theorem on the RHS of (3.4.16) to exchange $\lim_{\varepsilon\to0}$ with the integral over r_1. Indeed, the integral over y is bounded by a constant independent of r_1 and ε and has a pointwise limit as $\varepsilon\to0$ because ϕ_{EV} is a Schwartz function. The integrability in r_1 is guaranteed by $r_1\mapsto r_1^{l-1}\mathcal{M}^{-1}\tilde{g}_l(r_1)\in\mathrm{L}^1(\mathbb{R}_+)$. Finally, we arrive at the claim

$$\lim_{\varepsilon\to0}\mathcal{M}\left[\phi_{\mathrm{EV}}\cdot\mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon\cdot\tilde{g}_l]\right](l)=\mathcal{M}\left[\phi_{\mathrm{EV}}\cdot\lim_{\varepsilon\to0}\mathcal{M}_*^{-1}[\tilde{\zeta}_\varepsilon\cdot\tilde{g}_l]\right](l).\tag{3.4.17}$$

Thus, keeping $\lim_{\varepsilon\to0}$ outside of all the integrals (LHS of the above equation) and going on with our original aim of proving Theorem 3.3.1, we can now interchange the integrals over r_1, s_l and a as we like since the absolute integrability in s_l is given by the auxiliary function ζ, the one in r_1 by ϕ_{EV} and the one in a by f_{SV}. Setting $s_l=l+it$, with $t\in\mathbb{R}$, we note that the last term in (3.4.8) is not a problem because

$$\begin{aligned}&\left|\frac{\left(\prod_{j=0}^{n-1}j!\right)\det\left[a_b^{l+it-1},\ a_b^{\tau_c(l)-1}\right]_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-l-it)}\right|\\&=\left|\int_{\mathrm{U}(n)}\left(\frac{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l)\times(n-l)}\right)}{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l-1)\times(n-l-1)}\right)}\right)^{it}d\mu_H(U)\right|\leqslant1,\end{aligned}\tag{3.4.18}$$

where $(U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{p\times p}$ is the upper left $p\times p$ block and $d\mu_H(U)$ is the normalised Haar measure of the unitary group. The group integral in the second line is a particular case of the Gelfand-Naimark integral (2.3.13) which is, evidently, an integral of a bounded integrand over the unitary group.

To derive (3.4.18), one can look at the LHS of (3.4.18) and notice that the ratio

of determinants is invariant under permutations of the components of the vector $(0,\dots,l-2,s_l-1,l,\dots,n-1)$. One can then choose the following ordering $s'=(s'_1,\dots,s'_n)=(n-1,\dots,l,s_l-1,l-2,\dots,0)$, to apply (2.3.13). The computation of the generalised power (2.3.12) simplifies nicely as all but two terms are equal to 1,

$$\begin{aligned}\left|U\,\mathrm{diag}(a)U^{-1}\right|^{s'} = \det\Big(\big(U\,\mathrm{diag}(a)U^{-1}\big)_{(n-l-1)\times(n-l-1)}\Big)^{s'_{n-l-1}-s'_{n-l}-1}\\ \times\det\Big(\big(U\,\mathrm{diag}(a)U^{-1}\big)_{(n-l)\times(n-l)}\Big)^{s'_{n-l}-s'_{n-l+1}-1}.\end{aligned}\tag{3.4.19}$$

where $s'_{n-l-1}-s'_{n-l}-1=l-s_l=-it$ and $s'_{n-l}-s'_{n-l+1}-1=s_l-l=it$.

Moving forward, having interchanged the limit $\varepsilon\to 0$ with the r_1 integral, we then interchange the t and r_1 integrals with the a integral to slightly shift the contour of $t\to t+i\delta$ with a $\delta>0$, i.e.,

$$\begin{aligned}&\mathbb{E}\bigg[\frac{1}{n!\,n}\sum_{\substack{\sigma\in\mathrm{S}_n\\1\leqslant l\leqslant n}}\phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\bigg]\\ &=\frac{\big(\prod_{j=0}^{n-1}j!\big)}{n!\,n}\sum_{l=1}^{n}\lim_{\varepsilon\to0}\int_{\mathbb{R}_+^n}da\,f_{\mathrm{SV}}(a)\sum_{\sigma\in\mathrm{S}_n}\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})\int_0^\infty dr_1\,\phi_{\mathrm{EV}}(r_1)\\ &\quad\times\int_{-\infty}^{\infty}\frac{dt}{2\pi}\zeta(\varepsilon(t+i\delta))r_1^{-1+\delta-it}\frac{\det\Big[a_b^{l-1-\delta+it},\ a_b^{\tau_c(l)-1}\Big]_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-l+\delta-it)}.\end{aligned}\tag{3.4.20}$$

The shift is allowed as the integrand in t is an entire function and it drops off in the real direction at least like t^{-2} because of ζ.

The shift allows us to interchange the r_1 and t integrals so that we can integrate by parts twice in r_1,

$$\int_0^\infty dr_1\,\phi_{\mathrm{EV}}(r_1)r_1^{-1+\delta-it}=\frac{1}{(\delta-it)(1+\delta-it)}\int_0^\infty dr_1\,\phi''_{\mathrm{EV}}(r_1)r_1^{1+\delta-it}\tag{3.4.21}$$

which creates an additional decrease in t for $t\to\infty$. That decrease will now take over the job of the auxiliary function ζ. Since the integrability in a might be an issue because of the shift, we take the limit $\delta\to0$ which gives $-1/2$ times the residue at $t=-i\delta=0$ and a Cauchy principal value integral, denoted $⨍$, with a simple pole

at $t = 0$. This leads to

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{1}{n!\,n}\sum_{\substack{\sigma\in \mathrm{S}_n\\ 1\leqslant l\leqslant n}}\phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Bigg]\\
&=\frac{\left(\prod_{j=0}^{n-1} j!\right)}{n!\,n}\sum_{l=1}^{n}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a)\sum_{\sigma\in \mathrm{S}_n}\phi_{\mathrm{SV}}(a_{\sigma(1)},\dots,a_{\sigma(k)})\int_0^\infty dr_1\,\phi''_{\mathrm{EV}}(r_1)r_1\\
&\quad\times\left[\frac{1}{2(\prod_{j=0}^{n-1} j!)}-\mathcal{I}\right],
\end{aligned}\tag{3.4.22}$$

with the Cauchy principal value integral

$$\mathcal{I}:=\fint_{-\infty}^{\infty}\frac{dt}{2\pi it(1-it)}\zeta(\varepsilon t)r_1^{-it}\frac{\det\left[a_b^{l+it-1},\ a_b^{\tau_c(l)-1}\right]_{\substack{b=1,\dots,n\\ c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-l-it)}.\tag{3.4.23}$$

We can get back to an ordinary integral without principal value when defining the function

$$\Upsilon_l(t;r_1;a):=\frac{1}{t\Delta_n(a)\Delta_{n-1}(\tau(l))}\,\mathrm{Im}\left\{\frac{\det\left[r_1^{-it}a_b^{l+it-1},\ a_b^{\tau_c(l)-1}\right]_{\substack{b=1,\dots,n\\ c=1,\dots,n-1}}}{(1-it)\prod_{j=1}^{n-1}(\tau_j(l)-l-it)}\right\}\tag{3.4.24}$$

and note that, as $\mathcal{I}$ is real, and due to the symmetry $\zeta(\varepsilon t)=\zeta(-\varepsilon t)$, one has

$$\mathcal{I}=\frac{1}{2}(\mathcal{I}+\overline{\mathcal{I}})=\int_{-\infty}^{\infty}\frac{dt}{2\pi}\zeta(\varepsilon t)\Upsilon_l(t;r_1;a),\tag{3.4.25}$$

where we have used the fact $\mathrm{Re}\{-iz\}=\mathrm{Im}\{z\}$. The function $\Upsilon_l(t;r_1;a)$ has no pole at $t=0$ and is absolutely integrable in t as it behaves like $O(1/t^2)$ for $t\to\pm\infty$ for all $a\in\mathbb{R}_+^k$ and $r_1\in\mathbb{R}_+$.

The full integrand on the RHS of (3.4.22) is now bounded from above by the integrable function $f_{\mathrm{SV}}(a)|r_1\phi''_{\mathrm{EV}}(r_1)\Upsilon_l(t;r_1;a)|$ independent of ε since the rest is bounded by a constant while the pointwise limit $\varepsilon\to 0$ is clear. Thence, we can carry out this limit via Lebesgue's dominated convergence theorem and arrive at

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{1}{n!\,n}\sum_{\substack{\sigma\in \mathrm{S}_n\\ 1\leqslant l\leqslant n}}\phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Bigg]\\
&=\int_{\mathbb{R}_+^n} da\, f_{\mathrm{SV}}(a)\phi_{\mathrm{SV}}(a_1,\dots,a_k)\int_0^\infty dr_1\,\phi''_{\mathrm{EV}}(r_1)r_1\left[\frac{1}{2}-\frac{\prod_{j=0}^{n-1} j!}{n}\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\Upsilon_l(t;r_1;a)\right],
\end{aligned}\tag{3.4.26}$$

where we have made use of the permutation invariance of the integrand in a which yields an $n!$ factor. To figure out the analytic properties of the function in the bracket

it is helpful to plug in the Gelfand-Naimark integral (3.4.18) and interchange the integral over the unitary matrix U with the t-integral which can be carried out with standard residue theorem techniques yielding

$$\begin{aligned} & r_1\left[\frac{1}{2}-\frac{\left(\prod_{j=0}^{n-1} j!\right)}{n}\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\Upsilon_l(t;r_1;a)\right] \\ =&\frac{1}{n}\sum_{l=1}^{n}\int_{\mathrm{U}(n)}d\mu_H(U)\left[r_1-\frac{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l)\times(n-l)}\right)}{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l-1)\times(n-l-1)}\right)}\right] \\ &\times\Theta\left(r_1-\frac{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l)\times(n-l)}\right)}{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l-1)\times(n-l-1)}\right)}\right). \end{aligned} \tag{3.4.27}$$

This function is evidently once piecewise differentiable and continuous in r_1 so we can proceed with an integration by parts,

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{n!\,n}\sum_{\substack{\sigma\in\mathrm{S}_n\\1\leqslant l\leqslant n}}\phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\right] \\ &=-\int_{\mathbb{R}_+^n}da\int_0^\infty dr_1\,f_{\mathrm{SV}}(a)\phi_{\mathrm{SV}}(a_1,\dots,a_k)\phi'_{\mathrm{EV}}(r_1)\Xi(r_1;a) \end{aligned} \tag{3.4.28}$$

with

$$\begin{aligned} \Xi(r_1;a):=&\frac{1}{n}\sum_{l=1}^{n}\int_{\mathrm{U}(n)}d\mu_H(U)\Theta\left(r_1-\frac{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l)\times(n-l)}\right)}{\det\left((U\,\mathrm{diag}(a_1,\dots,a_n)U^\dagger)_{(n-l-1)\times(n-l-1)}\right)}\right) \\ =&\frac{\left(\prod_{j=0}^{n-1} j!\right)}{n}\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{dt\,r_1^{-it+1/2}}{2\pi(it-1/2)}\frac{\det\left[a_b^{l-3/2+it},\ a_b^{\tau_c(l)-1}\right]_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-l+1/2-it)}. \end{aligned} \tag{3.4.29}$$

The first expression, in terms of a group integral, makes it immediate that it does not cause any integrability issues in a and r_1. The latter integral has to be understood in the sense of Cauchy's principal value integral if all squared singular values a are equal to each other, and is absolutely convergent otherwise.

We are now ready to finish the proof. The function f_{SV} is a density with respect to the Lebesgue measure and the event of two singular values being equal to each other has Lebesgue measure 0. Hence, we can consider the r_1 integral in terms of fixed pairwise distinct square singular values. This tells us that the integrand in (3.4.29) behaves like $1/t^n$ for $t\to\infty$ and vanishes at $r_1=0$. For $n>2$ this is continuously

differentiable which allows us to integrate by parts once more, i.e.,

$$\begin{aligned}&\mathbb{E}\Bigg[\frac{1}{n!\,n}\sum_{\substack{\sigma\in S_n\\1\leqslant l\leqslant n}}\phi_{\mathrm{EV}}(r_l(X))\phi_{\mathrm{SV}}(a_{\sigma(1)}(X),\dots,a_{\sigma(k)}(X))\Bigg]\\&=\int_{\mathbb{R}_+^n}da\int_0^\infty dr_1\, f_{\mathrm{SV}}(a)\phi_{\mathrm{SV}}(a_1,\dots,a_k)\phi_{\mathrm{EV}}(r_1)\Xi'(r_1;a)\end{aligned}\tag{3.4.30}$$

with

$$\Xi'(r_1;a)=\frac{\left(\prod_{j=0}^{n-1}j!\right)}{n}\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{dt}{2\pi}r_1^{-it-1/2}\frac{\det\left[a_b^{l-3/2+it},\ a_b^{\tau_c(l)-1}\right]_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(a)\Delta_{n-1}(\tau(l))\prod_{j=1}^{n-1}(\tau_j(l)-l+1/2-it)}.\tag{3.4.31}$$

The last step is to set back $s=l+it$ and shift $s\mapsto s+1/2$ such that $s\in\mathcal{C}_l=l+i\mathbb{R}$, which is allowed due to the integrand being entire and dropping off rapidly enough. For $k=n>2$, this immediately yields the claim of the theorem. For $k<n$ we only integrate out the variables $a_{k+1},\dots,a_n$. This finishes the proof of the main statement of Theorem 3.3.1.

We note that the statement of the limit to degenerate $a_1,\dots,a_k$ is immediate as the integral over t converges at least conditionally as long as not all singular values are equal to each other. In the latter case when $a_1=\ldots=a_n$, the statement of the weak convergence becomes also clear as we have

$$\Xi(r_1;a)=\Theta(r_1-a_1)\tag{3.4.32}$$

which implies

$$-\int_0^\infty dr_1\,\phi'_{\mathrm{EV}}(r_1)\Xi(r_1;a)=\phi_{\mathrm{EV}}(a_1),\tag{3.4.33}$$

meaning the $1,n$-point measure comprises a Dirac delta function $\delta(r_1-a_1)$ and does not have a density anymore.

3.4.2 Proof of Corollary 3.3.2

We start with the result (3.3.1) for $k=n$, i.e.,

$$f_{1,n}(r;a_1,\dots,a_n)=\frac{1}{n}\left(\prod_{p=0}^{n-1}p!\right)\sum_{j=1}^{n}\int_{\mathcal{C}_j}\frac{ds}{2\pi i}r^{j-1-s}f_{\mathrm{SV}}(a)\frac{\det\begin{bmatrix}a_b^{s-1}\\a_b^{\tau_c(j)-1}\end{bmatrix}_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(s,\tau(j))\Delta_n(a)}.\tag{3.4.34}$$

As we do not carry out any integration over a the factor $f_{\mathrm{SV}}(a)$ is a constant in the remaining integration variable s and summing index j. Thus, the division by $f_{\mathrm{SV}}(a)$ immediately yields the first line of (3.3.2).

To get the second result, we first note that the integrand is entire in s, as the apparent $(n-1)$ simple poles cancel with the zeros generated by the determinant in the numerator. Therefore, we can shift the contour $\mathcal{C}_j$ to the imaginary axis $\mathcal{C}_0 = i\mathbb{R}$. Next, we evaluate the Vandermode determinant

$$\frac{1}{\Delta_n(s,\tau(j))} = \frac{(j-1)!\,(n-j)!}{\prod_{l=1}^{n-1} l!}\frac{j-s}{\prod_{l=1}^{n}(l-s)}. \tag{3.4.35}$$

and rewrite $(j-s)r^{j-1-s} = \partial_r r^{j-s}$. The derivative can be swapped with the integral over s because of the absolute integrability in s and holomorphicity in r and s (along $\mathcal{C}_0$) of the integrand. This means we consider

$$\rho_{\mathrm{EV}}(r|a) = \frac{1}{n}\partial_r \sum_{j=1}^{n}(j-1)!\,(n-j)! \int_{\mathcal{C}_0}\frac{ds}{2\pi i}\frac{\det\begin{bmatrix}(a_b/r)^{s-1}\\(a_b/r)^{\tau_c(j)-1}\end{bmatrix}_{\substack{b=1,\dots,n\\c=1,\dots,n-1}}}{\Delta_n(a/r)\prod_{l=1}^{n}(l-s)}, \tag{3.4.36}$$

where a/r is a shorthand notation for the vector $(a_1/r,\dots,a_n/r)$. The sum over j can be understood as a Laplace expansion in the first column of an $(n+1)\times(n+1)$ determinant, while the integral over s can be pulled into the first row which can then be computed by closing the contour either around the positive real line for $r > a_b$ or the negative real line for $a_b > r$, i.e.,

$$\int_{\mathcal{C}_0}\frac{ds}{2\pi i}\frac{(a_b/r)^{s-1}}{\prod_{l=1}^{n}(l-s)} = \Theta(r-a_b)\sum_{l=1}^{n}\frac{(-a_b/r)^{l-1}}{(l-1)!(n-l)!} = \frac{(1-a_b/r)^{n-1}}{(n-1)!}\Theta(r-a_b) \tag{3.4.37}$$

with Θ the Heaviside step function. We arrive at

$$\rho_{\mathrm{EV}}(r|a) = \frac{1}{n}\partial_r \frac{\det\left[\begin{array}{c|c}0 & (1-a_b/r)^{n-1}\Theta(r-a_b)\\ \hline (-1)^c\dfrac{(c-1)!(n-c)!}{(n-1)!} & (a_b/r)^{c-1}\end{array}\right]_{b,c=1}^{n}}{\Delta_n(a/r)}. \tag{3.4.38}$$

After pulling the factors $(-1)^{c-1}\frac{(c-1)!(n-c)!}{(n-1)!}$ out of the numerator and combining those with the denominator we find the second line of (3.3.2).

3.4.3 Proof of Theorem 3.3.7

To prove Theorem 3.3.7, we start from the following expression of the conditional level density of the squared radii

$$\rho_{\mathrm{EV}}(r|a) = \frac{\partial_r \det \left[\begin{array}{c|c} 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) \\ \hline \frac{(c-1)!(n-c)!}{n!}(-r)^{c-1} & a_b^{c-1} \end{array}\right]_{b,c=1}^{n}}{\Delta_n(a)}, \tag{3.4.39}$$

where b is a column index and c a row index. This equation is a direct consequence of (3.3.2). We multiply it by the polynomial ensemble density f_{SV} given by (2.1.18). The Vandermonde determinants $\Delta_n(a)$ in the numerator and denominator cancel each other and we integrate over $a_{k+1},\dots,a_n$ to arrive at the following expression of the $1,k$-point correlation function

$$\begin{aligned} f_{1,k}(r;a_1,\dots,a_k) = &\int_{\mathbb{R}_+^{n-k}} da_{k+1}\cdots da_n \frac{\det\left[w_{k-1}(a_j)\right]_{j,k=1}^{n}}{n!\,\det\left[\mathcal{M}w_{k-1}(j)\right]_{j,k=1}^{n}} \\ &\times \partial_r \det\left[\begin{array}{c|c} 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) \\ \hline \frac{(c-1)!(n-c)!}{n!}(-r)^{c-1} & a_b^{c-1} \end{array}\right]_{b,c=1}^{n}. \end{aligned} \tag{3.4.40}$$

Since $n>2$ the integrand as well as its derivative in r are continuous and integrable due to the weight functions. Thence, we can pull the derivative in r in front of the integral. Furthermore, we can rewrite

$$\frac{(c-1)!(n-c)!}{n!}(-r)^{c-1} = \int_0^\infty \frac{dt}{(1+t)^{n+1}}(-rt)^{c-1}, \tag{3.4.41}$$

which allows us to recombine the monomials to the monic polynomials $\{p_j\}_{j=0,\dots,n-1}$ which are biorthogonal with respect to the weights $\{q_j\}_{j=0,\dots,n-1}$ that span the vector space $\mathrm{span}\{w_0,\dots,w_n\}$, see (2.1.25). Thus, one can compute the integral over $a_{k+1},\dots,a_n$,

$$\begin{aligned} f_{1,k}(r;a_1,\dots,a_k) = &\frac{1}{n!}\partial_r \int_{\mathbb{R}_+^{n-k}} da_{k+1}\cdots da_n \int_0^\infty \frac{dt}{(1+t)^{n+1}} \det\left[q_{d-1}(a_e)\right]_{d,e=1}^{n} \\ &\times \det\left[\begin{array}{c|c} 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) \\ \hline p_{c-1}(-rt) & p_{c-1}(a_b) \end{array}\right]_{b,c=1}^{n}, \end{aligned} \tag{3.4.42}$$

with the help of a generalisation of Andréief's identity given in Proposition 3.3.6. We are allowed to interchange the two sets of integrals due to the absolute integrability.

We arrive at

$$f_{1,k}(r;a_1,\dots,a_k) = \frac{(n-k)!}{n!}\partial_r \int_0^\infty \frac{dt}{(1+t)^{n+1}} \times \det \left[\begin{array}{c|c|c} 0 & 0 & q_{d-1}(a_e) \\ \hline 0 & -\left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b) & -\int_0^r dv\; q_{d-1}(v)\left(1-\frac{v}{r}\right)^{n-1} \\ \hline p_{c-1}(-rt) & p_{c-1}(a_b) & \delta_{cd} \end{array}\right], \tag{3.4.43}$$

where $b = 1,\dots,k$ and $d = 1,\dots,n$ are column indices and $c = 1,\dots,n$ and $e = 1,\dots,k$ are row indices. We have employed the bi-orthogonality (2.1.25) in the lower right block of the determinant.

Next, we exploit the standard identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det\left(A - BD^{-1}C\right) \tag{3.4.44}$$

and identify the kernel (2.1.24) which results in

$$f_{1,k}(r;a_1,\dots,a_k) = \frac{(n-k)!}{n!}\partial_r \int_0^\infty \frac{dt}{(1+t)^{n+1}} \times \det \left[\begin{array}{c|c} -K_n(a_e,-rt) & -K_n(a_e,a_b) \\ \hline \int_0^r dv\; K_n(v,-rt)\left(1-\frac{v}{r}\right)^{n-1} & \Omega(r,a_b) \end{array}\right]_{b,e=1}^{k} \tag{3.4.45}$$

with

$$\Omega(r,a_b) := \int_0^r dv\; K_n(v,a_b)\left(1-\frac{v}{r}\right)^{n-1} - \left(1-\frac{a_b}{r}\right)^{n-1}\Theta(r-a_b). \tag{3.4.46}$$

After pulling out the minus sign from the first k rows and then interchanging those with the last row, the signs are cancelling. Moreover, we can pull the integral over v out of the determinant by introducing a Dirac delta function yielding the first result of Theorem 3.3.7.

When carrying out the derivative in r we arrive at the second expression. To achieve this, one can do a Laplace expansion along the first column (or first row) in the first expression (3.3.10) and integrate by parts in t, using the fact that

$$\partial_r K_n(x,-rt) = \frac{t}{r}\partial_t K_n(x,-rt). \tag{3.4.47}$$

The differentiability is guaranteed and the boundary terms vanish because of $n > 2$ and the factor of t from the latter equation.

3.4.4 Proof of Corollary 3.3.9

To prove this proposition we need the following lemma.

Lemma 3.4.2 *Let K_n be the kernel (2.1.24) of a polynomial ensemble. Then,*

$$\int_0^\infty K_n(x,y)x^k dx = y^k \qquad \text{for all } k = 0,\dots,n-1. \tag{3.4.48}$$

Proof of Lemma 3.4.2. Let, $\{p_b\}_{b=0}^{n-1}$, with p_b a polynomial of degree b, and $\{q_b\}_{b=0}^{n-1}$ be the bi-orthonormal system given by the polynomial ensemble, i.e.,

$$\int_0^\infty dx\ q_b(x)p_c(x) = \delta_{c,b}. \tag{3.4.49}$$

Any polynomial f of degree less than n has a unique decomposition in the basis $\{p_b\}_{b=0}^{n-1}$, namely

$$f(x) = \sum_{b=0}^{n-1}\left(\int_0^\infty dt\, q_b(t)f(t)\right)p_b(x). \tag{3.4.50}$$

Taking $f(y) = y^k$ for a $k \in [\![0, n-1]\!]$ the result follows. ■

Proof of Corollary 3.3.9. Starting from the second line of Eq. (3.3.10) and subtracting $f_{1,0}(r)f_{0,k}(a_1,\dots,a_k)$ we find

$$\begin{aligned}
&\mathrm{cov}_{1,k}(r;a_1,\dots,a_k)\\
&= \frac{(n-k)!}{(n-1)!}\int_0^\infty dt\int_0^r \frac{dv}{v}\varphi_n\left(\frac{v}{r},t\right)\det\left[\begin{array}{c|c} 0 & K_n(v,a_c)-\delta(v-a_c)\\ \hline K_n(a_b,-rt) & K_n(a_b,a_c)\end{array}\right]_{b,c=1}^{k}\\
&= -\frac{(n-k)!}{(n-1)!}\sum_{l,m=1}^{k}(-1)^{l+m}\det[K_n(a_b,a_c)]_{\substack{b\in[\![1,k]\!]\setminus\{l\}\\ c\in[\![1,k]\!]\setminus\{m\}}}\\
&\quad\times\underbrace{\int_0^\infty dt\ K_n(a_l,-rt)\int_0^r\frac{dv}{v}\varphi_n\left(\frac{v}{r},t\right)[K_n(v,a_m)-\delta(v-a_m)]}_{=:-\mathbf{C}_n(r;a_l,a_m)}.
\end{aligned} \tag{3.4.51}$$

We can rewrite the sum by introducing an auxiliary parameter μ and understand the expression as a derivative of a determinant, using Jacobi's formula (A.2.4), i.e.,

$$\mathrm{cov}_{1,k}(r;a_1,\dots,a_k) = \frac{(n-k)!}{(n-1)!}\partial_\mu \det\left[\ K_n(a_b,a_c)+\mu\,\mathbf{C}_n(r;a_b,a_c)\ \right]_{b,c=1}^{k}\Big|_{\mu=0}. \tag{3.4.52}$$

The expression (3.3.18) is left to show. After performing the change of variables $-t = \tau/(\tau-1)$ and employing the definition (3.3.11) and applying the Dirac delta

function, we have

$$\begin{aligned}\mathbf{C}_n(r;a_1,a_2) &= -\int_0^\infty dt\, K_n(a_1,-rt)\left[\int_0^r \frac{dv}{v}\varphi_n\left(\frac{v}{r},t\right)K_n(v,a_2) - \frac{\Theta(r-a_2)}{a_2}\varphi_n\left(\frac{a_2}{r},t\right)\right] \\ &= \int_0^\infty dv\; T(r,v,a_1)K_n(v,a_2) - T(r,a_1,a_2),\end{aligned} \tag{3.4.53}$$

with

$$T(r,v,a) := \frac{1}{n^2}\int_0^1 d\tau \frac{1}{v}h\left(\frac{r}{v},\tau\right)K_n\left(a,\frac{r\tau}{\tau-1}\right), \tag{3.4.54}$$

where

$$h(x,\tau) = -n\left[\Psi_0(x^{-1})(1-\tau)^{n-1} + (n+1)\Psi_1(x^{-1})\tau(1-\tau)^{n-1}\right]\Theta(1-x), \tag{3.4.55}$$

with Ψ_0 and Ψ_1 defined in (3.3.19). Using the integral representation of the kernel (2.1.37) in (3.4.54), we obtain

$$T(r,v,a) = \frac{1}{n}\int_0^1 \frac{d\tau}{v}h\left(\frac{r}{v},\tau\right)\int_0^1 dt q_n(at)p_{n-1}\left(\frac{rt\tau}{\tau-1}\right). \tag{3.4.56}$$

The absolute integrability in t is given by $t \mapsto q_n(at)$ while the one in τ follows from $(1-\tau)^{n-1}|p_{n-1}(rt\tau/(\tau-1))| < \infty$ for all $\tau \in [0,1]$. Thus, we can interchange the two integrals and find

$$\begin{aligned}T(r,v,a) = -\frac{1}{v}\Theta(v-r)\Bigg[&\Psi_0\left(\frac{v}{r}\right)\int_0^1 dt q_n(at)\int_0^1 d\tau p_{n-1}\left(\frac{rt\tau}{\tau-1}\right)(1-\tau)^{n-1} \\ &+ (n+1)\Psi_1\left(\frac{v}{r}\right)\int_0^1 dt q_n(at)\int_0^1 d\tau p_{n-1}\left(\frac{rt\tau}{\tau-1}\right)\tau(1-\tau)^{n-1}\Bigg].\end{aligned} \tag{3.4.57}$$

To shorten the notation, we define

$$P_\gamma(x) := (n+1)^\gamma\int_0^1 d\tau p_{n-1}\left(\frac{x\tau}{\tau-1}\right)\tau^\gamma(1-\tau)^{n-1},\;\; H_\gamma(x,y) := \int_0^1 dt P_\gamma(xt)q_n(yt) \tag{3.4.58}$$

for $\gamma = 0,1$. In terms of these functions, we have

$$T(r,v,a) = -\frac{1}{v}\Theta(v-r)\sum_{\gamma=0,1}\Psi_\gamma\left(\frac{v}{r}\right)H_\gamma(r,a) \tag{3.4.59}$$

and

$$\int_0^\infty T(r,v,a)K_n(v,a)dv = -n\int_1^\infty \frac{du}{u}\int_0^1 dt\sum_{\gamma=0,1}\Psi_\gamma(u)H_\gamma(r,a)q_n(rut)p_{n-1}(at), \tag{3.4.60}$$

where we substituted $v = ru$.

To interchange the integrals in (3.4.60), we need to be more careful as the integration is not absolutely convergent when combined in u and t as the weight function

q_n, which usually guarantees the convergence, depends on the product ut which is troublesome when t becomes small while u is large. The idea is to go back to the expression (2.1.24) of the kernel in terms of a sum and use the bi-orthonormality relations between p_k and q_j. We note that the function $u \mapsto u^{-1}\Psi_\gamma(u)$ is a polynomial in u of degree $n-1$ for both $\gamma = 0, 1$. Thence, it can be decomposed in terms of the polynomials $u \mapsto p_j(xu)$ with an arbitrary $x > 0$, i.e.,

$$u^{-1}\Psi_\gamma(u) = \sum_{k=0}^{n-1} a_k(x) p_k(xu). \tag{3.4.61}$$

The auxiliary variable x is important since the kernel comes with $x = r$. We can combine this with

$$\int_1^\infty \frac{du}{u}\Psi_\gamma(u)K_n(xu,y) = \int_0^\infty \frac{du}{u}\Psi_\gamma(u)K_n(xu,y) - \int_0^1 \frac{du}{u}\Psi_\gamma(u)K_n(xu,y). \tag{3.4.62}$$

Due to Lemma 3.4.2 we have then

$$\begin{aligned}\int_0^\infty \frac{du}{u}\Psi_\gamma(u)K_n(xu,y) &= \sum_{k=0}^{n-1} a_k(x) \int_0^\infty du\; p_k(xu)K_n(xu,y) \\ &= \sum_{k=0}^{n-1} a_k(x)\frac{p_k(y)}{x} = \frac{1}{y}\Psi_\gamma\left(\frac{y}{x}\right).\end{aligned} \tag{3.4.63}$$

The integrals in the remaining integration over $u, t \in [0,1]$ can be interchanged now, leading to

$$\int_0^1 \frac{du}{u}\Psi_\gamma(u)K_n(xu,y) = n\int_0^1 dt p_{n-1}(yt)\int_0^1 \frac{du}{u}\Psi_\gamma(u)q_n(xut). \tag{3.4.64}$$

We combine these considerations with

$$\int_0^\infty \frac{du}{u}\Psi_\gamma(u)q_n(xut) = 0 \tag{3.4.65}$$

and plug them into (3.4.60) to arrive at

$$\begin{aligned}\int_0^\infty T(r,v,a)K_n(v,a)dv = &-\sum_{\gamma=0,1}\frac{1}{a}\Psi_\gamma\left(\frac{a}{r}\right)H_\gamma(r,a) \\ &- n\int_0^1 dt \sum_{\gamma=0,1} H_\gamma(r,a)p_{n-1}(at)\int_1^\infty \frac{du}{u}\Psi_\gamma(u)q_n(rut).\end{aligned} \tag{3.4.66}$$

We define for $\gamma = 0, 1$ the functions

$$Q_\gamma(x) := -n\int_1^\infty \frac{du}{u}\Psi_\gamma(u)q_n(xu), \qquad V_\gamma(x,y) := \int_0^1 dt Q_\gamma(xt)p_{n-1}(yt), \tag{3.4.67}$$

so that the cross-covariance density (3.3.16) takes the desired form (3.3.17).

What is left is to compute Q_γ and P_γ using the explicit expressions (2.1.38) for p_{n-1} and q_n. For the function P_γ we start from its definition (3.4.58) and compute

$$P_\gamma(x) = (n+1)^\gamma \sum_{c=0}^{n-1} \binom{n-1}{c} \frac{x^c}{\tilde{w}_n(c+1)} \int_0^1 d\tau \tau^{\gamma+c} (1-\tau)^{n-1-c}. \tag{3.4.68}$$

Using the formula for the Beta function, we get

$$P_\gamma(x) = \frac{1}{n} \sum_{c=0}^{n-1} \frac{x^c}{\tilde{w}_n(c+1)} (c+1)^\gamma, \quad \gamma = 0, 1. \tag{3.4.69}$$

The resulting sum can be identified with the 1-point function of the squared eigenradii for Pólya ensembles, see [91, Lemma 4.1]. Explicitly,

$$P_0(x) = \frac{\rho_{\mathrm{EV}}(x)}{w_n(x)} \qquad \text{and} \qquad P_1(x) = \partial_x \left[x \frac{\rho_{\mathrm{EV}}(x)}{w_n(x)} \right]. \tag{3.4.70}$$

For the function Q_γ, we use the derivative formula (2.1.38) for q_n and plug it into the definition (3.4.67). For this aim, we consider the integral

$$\widehat{Q}_\gamma(x) := \int_1^\infty du\, q_n(xu)(u-1)^{n-2+\gamma}. \tag{3.4.71}$$

Changing the integration variable $v = xu$, we need to compute

$$\widehat{Q}_\gamma(x) = \frac{x^{-(n-1+\gamma)}}{n!} \int_x^\infty dv\, \partial_v^n [v^n w_n(v)](v-x)^{n-2+\gamma}. \tag{3.4.72}$$

We can now integrate by parts $n-2+\gamma$ times. As the boundary terms are vanishing, this yields

$$\widehat{Q}_0(x) = \frac{(-1)^{n-1}}{n(n-1)} [n w_n(x) + x \partial_x w_n(x)], \qquad \widehat{Q}_1(x) = \frac{(-1)^n}{n} w_n(x). \tag{3.4.73}$$

The functions Q_γ expressed in terms of $\widehat{Q}_\gamma$ are

$$\begin{aligned} Q_0(x) &= (-1)^{n-1} n [n\widehat{Q}_1(x) + (n-1)\widehat{Q}_0(x)] = x\partial_x w_n(x), \\ Q_1(x) &= (-1)^{n-2} n \widehat{Q}_1(x) = w_n(x). \end{aligned} \tag{3.4.74}$$

Plugging this into (3.4.67) concludes the proof. ■

Almost all the integrals involved in the formulation of Theorem 3.3.7 can be carried out, yielding an expression of the cross-covariance density which is very efficient for numerical evaluations. The only remaining integration is hidden in the incomplete Mellin transform of w_n. We will use Corollary 3.3.9 to prove the following corollary for $\mathrm{cov}(r; a) = \mathbf{C}_n(r; a, a)$.

Corollary 3.4.3 *Let $n \in \mathbb{N}$, $n > 2$. With the same assumptions and notations as in Theorem 3.3.7 and choosing a Pólya ensemble associated to the weight function w_n, the cross-covariance density can be cast into the form*

$$\begin{aligned}\mathrm{cov}(r;a) = \frac{\Theta(r-a)}{nar}\sum_{c=0}^{n-1}\frac{(r/a)^c}{\tilde{w}_n(c+1)}\tilde{q}_{n,a}(c+1)\left(1-\frac{a}{r}\right)^{n-2}\left[(n-c-1)\frac{a}{r}+c\right]\\ -\frac{1}{na}\sum_{c=0}^{n-1}\frac{(r/a)^c}{\tilde{w}_n(c+1)}\tilde{q}_{n,a}(c+1)\Bigg[w_n(r)p_{n-1}(a)\\ +\sum_{j=0}^{n-1}\binom{n-1}{j}(c-j)\frac{\tilde{w}_{n,r}(j+1)}{\tilde{w}_n(j+1)}\frac{(-a)^j}{r^{j+1}}\Bigg].\end{aligned} \tag{3.4.75}$$

We recall the definitions and notations of the Mellin transform (2.3.6) and the incomplete Mellin transform (2.3.7).

Proof of Corollary 3.4.3. Considering the proof of Corollary 3.3.9, we still need to compute V_γ and H_γ. Using the sum expression (3.3.22) of p_{n-1} and the explicit expression (3.4.74) of Q_γ with the change of variables $u = xt$, one gets

$$V_1(x,y) = \sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(-y/x)^j}{\tilde{w}_n(j+1)}\frac{\tilde{w}_{n,x}(j+1)}{x}, \tag{3.4.76}$$

while for V_0 it is

$$V_0(x,y) = \sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(-y/x)^j}{\tilde{w}_n(j+1)}\frac{1}{x}\int_0^x du\; u^{j+1}\partial_u w_n(u). \tag{3.4.77}$$

One can proceed with integration by parts to arrive at

$$V_0(x,y) = w_n(x)p_{n-1}(y) - \partial_y\left[yV_1(x,y)\right]. \tag{3.4.78}$$

Similarly to V_γ, one can also compute H_γ by exploiting (3.4.70) for P_γ and the derivative expression (3.3.22) of q_n. Then, we obtain for $\gamma = 0, 1$

$$H_\gamma(x,y) = \frac{1}{n!}\int_0^1 dt\sum_{c=0}^{n-1}\frac{(xt)^c}{\tilde{w}_n(c+1)}(c+1)^\gamma\,\partial^n_{(yt)}[(yt)^n w_n(yt)]. \tag{3.4.79}$$

Changing to $u = yt$, it is

$$H_\gamma(x,y) = \frac{1}{n!}\sum_{c=0}^{n-1}\frac{(x/y)^c}{\tilde{w}_n(c+1)}(c+1)^\gamma\frac{1}{y}\overbrace{\int_0^y du\; u^c\partial_u^n[u^n w_n(u)]}^{=n!\,\tilde{q}_{n,y}(c+1)}. \tag{3.4.80}$$

When plugging these explicit expressions for H_γ and V_γ into (3.3.17) one gets the claim of Corollary 3.4.3. ■

We would like to point out that the remaining integral is nothing else than the incomplete Mellin transform of q_n which we can express in terms of a sum after integrating by parts and collecting all boundary terms,

$$n!\,\tilde{q}_{n,y}(c+1) = \int_0^y u^c \partial_u^n [u^n w_n(u)] du = \sum_{p=0}^{c} \binom{c}{p} p!(-1)^p y^{c-p} \partial_y^{n-1-p} [y^n w_n(y)]. \tag{3.4.81}$$

Remark 3.4.4 *Equation (3.4.75) can also be cast into the factorized form*

$$\begin{aligned} \operatorname{cov}(x;y) =& \frac{1}{nxy} \sum_{c,j=0}^{n-1} \binom{n-1}{j} (-1)^j \left(\frac{y}{x}\right)^{j-c} \frac{\tilde{q}_{n,y}(c+1)}{\tilde{w}_n(c+1)} \\ &\times \Bigg[(c+1) \left(\Theta(x-y) - \frac{\tilde{w}_{n,x}(j+1)}{\tilde{w}_n(j+1)} \right) \\ &\quad - \Theta(x-y) \frac{x-ny}{x-y} + \frac{(j+1)\tilde{w}_{n,x}(j+1) - x^{j+1} w_n(x)}{\tilde{w}_n(j+1)} \Bigg], \end{aligned} \tag{3.4.82}$$

where we used the Newton binomial formula for the term $(1-y/x)^{n-2}$ *and the identity*

$$\int_0^x u^{j+1} \partial_u w_n(u) du = x^{j+1} w_n(x) - (j+1)\tilde{w}_{n,x}(j+1). \tag{3.4.83}$$

3.4.5 Proof of Corollary 3.3.10

For $n = 2$, the explicit formula in Proposition 3.2.4, clearly shows that the $1,1$- and $1,2$-point correlation functions are discontinuous along the lines $r = a_1$ and $r = a_2$ which is reflected in the Heaviside step functions.

For $n > 2$, we consider the expression of the $1,k$-point function given in Theorem 3.3.7. It is clear that ρ_{EV}—corresponding to the $(1,1)$ entry of the determinant—and the kernel K_n are continuous and smooth on the support σ for a Pólya ensemble because of their dependence on the smooth weight function $w_n \in \mathrm{C}^\infty(\sigma)$, see (2.1.37) and (2.1.47). The smoothness of the kernel K_n in its first argument is due to the compactness of the second integral expression in (2.1.37) and integrability of q_n on the support σ so that the Leibniz integral rule applies.

In summary, the reduced differentiability of $f_{1,k}$ must be inherited from the function

$$(r,v,a) \mapsto \frac{\Theta(r-v)}{v} \int_0^\infty dt\, \varphi_n\left(\frac{v}{r}, t\right) K_n(a, -rt) \tag{3.4.84}$$

see (3.4.54). Since K_n is a polynomial of degree $n-1$ in its second entry the integral over t gives a smooth function on the support σ in all three variables r, v and t. Furthermore, this integral does not vanish at almost all $v = r = a$ because of

$$\begin{aligned}
&\lim_{v,r\to a} \int_0^\infty dt\,(1+t)^{-(n+2)} \left[\left(1-\frac{v}{nr}\right)(1+t) - \left(1-\frac{v}{r}\right)\left(1+\tfrac{1}{n}\right)\right] K_n(a,-rt) \\
=&\frac{n-1}{n}\int_0^\infty dt\,(1+t)^{-(n+1)} \int_0^1 ds\; q_n(as) p_{n-1}(-ast) \\
=&\frac{n-1}{n}\int_0^1 ds\; q_n(as) P_0(as) \\
=&\frac{n-1}{an}\int_0^a ds\; q_n(s)\frac{\rho_{\mathrm{EV}}(s)}{w_n(s)}.
\end{aligned} \tag{3.4.85}$$

In the first equality we have exploited the integral representation (2.1.37) of the kernel, and in the second equality we could interchange the two integrals as they are absolutely integrable due to the polynomial nature of p_{n-1}. Additionally, we have employed Eqs. (3.4.58) and (3.4.70). As can be readily checked, the derivative in a of a times this integral is equal to $(n-1)q_n(a)\rho_{\mathrm{EV}}(a)/w_n(a)$ which is evidently non-linear implying a non-constant behaviour of this integral when $a \in \sigma$.

In conclusion of this discussion, the function (3.4.84) is $(n-3)$-times continuous differentiable at $r = a$ and its $(n-2)$nd derivative is discontinuous along this line because of the factor $\Theta(a-r)\,(a/r-1)^{n-2}$.

The question is whether the integral

$$\int_0^r \frac{dv}{v}\int_0^\infty dt\,\varphi_n\left(\frac{v}{r},t\right) K_n(a_1,-rt)K_n(v,a_2) \tag{3.4.86}$$

may change this differentiability. The point is that the integrand is smooth in $(r,a_1,a_2) \in \sigma^3$ and $(v,t) \in (0,r)\times\mathbb{R}_+$ and absolutely integrable in v and t. Thus, the Leibniz integral rule tells us that after integrating v and t it remains smooth in r and a. In summary, this tells us that the cross-covariance density (3.3.16) is a sum of a smooth function and an $(n-3)$-times continuous differentiable function on σ^{k+1} which has been the statement of the corollary.

3.5 Summary and discussion

In this chapter we studied $1,k$-point correlation functions (cf. Definition 2.3.7 and Definition 2.3.11) between the singular values and eigenvalues of arbitrary bi-invariant complex square matrices and found results for the $1,k$-point correlation function $f_{1,k}$, see Theorem 3.3.1, and for the 1-point function ρ_{EV} of the eigenradii, see Corollary 3.3.5. These formulas drastically simplify in the case the bi-unitarily invariant ensemble is a polynomial ensemble, see Remark 2.1.8, Theorem 3.3.7 and Remark 3.3.8. The obtained formula enables us to define and identify the $1,k$-cross-covariance density function between one eigenradius and k singular values,

which captures the interaction between the two kinds of variables. Note that the interaction could also be studied by defining analogues of cluster functions used in the physics literature [42, Eq.(5)], involving, in this case, the j,k-point correlation functions.

The simplification of the results increases further for Pólya ensembles [91]; see Corollary 3.3.9. Although the proof for $n = 2$ is very different, one can check that Corollary 3.3.9 agrees with Proposition 3.2.4 for an arbitrary polynomial ensemble, especially (3.2.14).

Let us underline that the only known formula for ρ_{EV} for bi-unitarily invariant ensembles in terms of quantities of the squared singular values f_{SV} was, so far, only for Pólya ensembles; see [91, Eq.(4.4)]. We found the generalisation (3.3.13) to polynomial ensembles which is a much larger class than Pólya ensembles.

The formulas (3.3.10) and (3.3.17) for the $1,k$-point correlation function were not known even for Pólya ensembles. Following Remark 3.3.8, they can be expressed in terms of the $1,k$-cross-covariance density.

A generalisation of our results for $f_{1,k}$ to an arbitrary j,k-point correlation function $f_{j,k}$ for j squared eigenradii and k squared singular values with $j > 1$ is certainly desirable, however, it will be challenging to obtain explicit results due to the Vandermonde determinant $\Delta_n(s)$ in the SEV transform (2.3.15). Indeed, this term couples all the poles in the computation of the n-fold complex integral and in general the regularisation function cannot be dropped. Another issue is that there is a trade-off between getting formulas for more eigenradii and having explicit and compact formulas as each fixed eigenradius comes with an additional complex contour integral.

Another goal of our study has been to find all conditional marginal measures of the eigenvalues when fixing the singular values. We have been successful for the level density of the eigenradii conditioned to fixed singular values, see Corollary 3.3.2. The n-point correlation function for the eigenradii conditioned to singular values is more complicated though we have a conjecture. Indeed, 2-point correlation function for $n = 2$ can be read off from Proposition 3.2.4 summarised in the following corollary.

Corollary 3.5.1 *For $n = 2$, the conditional probability measure of the squared eigenradii r_1, r_2 under the condition of the squared singular values a_1, a_2 is given by*

$$\begin{aligned} d\mu_{2,2}(r_1, r_2|a_1, a_2) =& \Theta\left(\max\{a_1, a_2\} - \max\{r_1, r_2\}\right) \Theta\left(\min\{r_1, r_2\} - \min\{a_1, a_2\}\right) \\ &\times \frac{r_1 + r_2}{2|a_1 - a_2|}\, \delta(r_1 r_2 - a_1 a_2) dr_1 dr_2. \end{aligned} \tag{3.5.1}$$

We recall that the Dirac distribution $\delta(r_1 r_2 - a_1 a_2)$ reflects the identity (1.1.6) while the Heaviside step functions encode the inequalities (1.1.8). Considering the SEV transform (2.3.15) one can conjecture that the conditional joint probability

density of the eigenvalues is given, in a distributional way, by

$$\begin{aligned} d\mu_{n,n}(z|a) =& \frac{\prod_{j=0}^{n-1} j!}{(n!)^2\pi^n}|\Delta_n(z)|^2 \lim_{\varepsilon\to 0}\int_{\mathcal{C}(n)}\left[\prod_{k=1}^{n}\frac{ds_k}{2\pi i}\zeta(\varepsilon\,\mathrm{Im}\{s_k\})\right] \\ &\times \mathrm{perm}[|z_b|^{-2s_c}]_{b,c=1}^{n}\frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}dz. \end{aligned} \tag{3.5.2}$$

The condition (1.1.6) is then still encoded in the integral over s and the limit $\varepsilon \to 0$. At the moment, one needs to take this formula with caution as a rigorous mathematical proof is missing for this conjecture because the limit $\varepsilon \to 0$ cannot exist pointwise but has to be understood in a distributional sense.

Reversing the conditional probability measure, meaning finding the measure for the squared singular by conditioning the eigenvalues, is expected to be more challenging when considering the inverse SEV transform, see [91, Eq. (3.4)].

Our results for polynomial ensembles and especially for Pólya ensembles open a pathway to study the large n limit for j,k-point correlation functions between eigenvalues and singular values. For instance, it allows to analyse finite n corrections of Feinberg-Zee's single ring theorem [48, 49, 66, 128]. The finite n analogue is encoded in Corollary 3.3.5 or more explicitly for polynomial ensembles in (3.3.13). In particular, formula (3.3.13) will enable us to understand how the asymptotics described by these theorems are approached.

Recently, a deformed Single Ring Theorem has been proven in [76]. It arises when the bi-unitary invariance is perturbed. Some kind of such perturbations might be also possible to study with our methods, as already pointed out in [91, Sec. 3.3].

There will also be interesting local spectral statistics of the cross-correlations between the complex eigenvalues and singular value. For instance the condition (1.1.10), implies non-trivial correlations around common edges. In the case of the common hard edge at the origin, it can already be seen in Figure 3.4.

Chapter 4

Hard edge asymptotics of correlation functions between singular values and eigenvalues at the origin

An idea which can be used once is a trick. If it can be used more than once it becomes a method.

GEORGE PÓLYA (1887–1985)

This chapter is based on the work [8].

4.1 Preamble

In this chapter, we build on the results of the previous one—where everything was carried out at fixed matrix size n—and now study the large-n limit of the collective behavior of the eigenvalues and singular values on a local scale. We will however focus on polynomial ensembles as for general bi-unitarily invariant ensembles, the result Theorem 3.3.1 is not really transparent regarding the large n limit. Indeed, in (3.3.1) the number of variables of integration increases with n, while the formulation (3.3.10) only involves a 2-fold integral and a fixed size determinant.

First, in section 4.2, we will look at the double scaling limit of both the $1, k$-point correlation function and the $1, k$-cross-covariance density around the origin for polynomial ensembles. This is interesting and non trivial, as most polynomial ensembles will have a hard edge at the origin. Indeed, unless there is a strong repulsion from the origin, the lower edge of the support will be a hard wall due to the definition of eigenradii and singular values, which are non negative. Moreover, the deterministic Weyl's inequalities [144] between eigenradii and singular values imply non trivial correlations between the two, where they share the same hard edge. The results are then made more explicit for Pólya ensembles. The main results of this section are Theorem 4.2.5, Theorem 4.3.8 and Corollary 4.3.15. We

apply the results to the example of the Muttalib-Borodin ensembles.

In section 4.6 we give the proofs and we then discuss the results and give some plots in section 4.7.

The principal challenge that had to be overcome in this chapter is the following. While hard-edge results for the singular values near the origin were already known for Pólya ensembles [89], these relied on assumptions imposed directly on the Pólya weight. Moreover, these results did not extend[1] to the eigenvalue setting, thereby requiring additional assumptions. To further generalize the hard-edge analysis to the broader and less structured class of polynomial ensembles, it was necessary to identify the appropriate conditions on the kernel of polynomial ensembles that would:

1. generalize (be implied by) the set of assumptions made for Pólya ensembles,
2. justify interchanging the limit $n \to \infty$ with the double integral involved in the formulation of Theorem 3.3.7,
3. be reasonably easy to check,
4. include, a priori, more ensembles than just Pólya ensembles.[2]

4.2 Polynomial ensembles

Let us start with some heuristics. When looking at Theorem 3.3.7, the double scaling limit (if existent) of the $1, k$-point correlation function will be given by scaling appropriately the squared eigenradius and the squared singular values and then taking the large n limit. One would expect the result to involve the double scaling limit of the kernels K_n and the double scaling limit of the φ_n; cf. subsection 2.2.2.

The goal is then to find the correct scalings. To begin, we examine (3.3.10) and first proceed with the change of variable $v \mapsto rv$. The arguments of φ_n do not depend on r or a anymore and one can see that the term $n^2\varphi_n(x/n, t/n)$ admits a pointwise limit

$$\lim_{n\to\infty} n^2\varphi_n\left(\frac{x}{n}, \frac{t}{n}\right) = x(t+x-1)e^{-x}e^{-t}. \tag{4.2.1}$$

This hints one needs to proceed with the change of integration variables $(v, t) \mapsto (v/n, t/n)$.

Now, let us assume that the appropriate scaling for the squared eigenradius is given by ν_n, thus we let $r \mapsto r/\nu_n$. It becomes clear that if the result involves the double scaling limit of the kernels, the correct scaling should be

$$\frac{1}{n\nu_n} K_n\left(\frac{x}{n\nu_n}, \frac{y}{n\nu_n}\right). \tag{4.2.2}$$

[1] A priori, these additional assumptions are required. It is conceivable that they may ultimately prove to be redundant; however, this has not yet been established and remains unclear.

[2] Unfortunately, this latter requirement has not been checked explicitly. We only show that the assumptions for Pólya ensembles partially imply the ones made for polynomial ensembles.

To be consistent, the correct scaling for the squared singular values is then $n\nu_n$, thus we set $(a_1, \dots, a_k) \mapsto \left(\frac{a_1}{n\nu_n}, \dots, \frac{a_k}{n\nu_n}\right)$. This surprising difference between the scaling of the squared eigenradius and squared singular values will be discussed later.

Accounting for the change of measure and multiplying by the appropriate global scaling, the correct scaling of the $1, k$-point correlation function between one squared eigenradius and k squared singular values should be

$$\begin{aligned}
&\frac{n^{k+1}}{\nu_n(n\nu_n)^k} f_{1,k}\left(\frac{r}{\nu_n}; \frac{a_1}{n\nu_n}, \dots, \frac{a_k}{n\nu_n}\right) \\
&\quad = \frac{n^k(n-k)!}{n!} \int_0^\infty dt \int_0^n \frac{dv}{v} n^2 \varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) \\
&\quad \times \det \begin{pmatrix} \frac{1}{n\nu_n} K_n\left(\frac{rv}{n\nu_n}, \frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n} K_n\left(\frac{rv}{n\nu_n}, \frac{a_c}{n\nu_n}\right) - \delta(rv - a_c) \\ \frac{1}{n\nu_n} K_n\left(\frac{a_b}{n\nu_n}, \frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n} K_n\left(\frac{a_b}{n\nu_n}, \frac{a_c}{n\nu_n}\right) \end{pmatrix}_{b,c=1}^k,
\end{aligned} \tag{4.2.3}$$

where we have used the scaling property of the Dirac delta function

$$\forall c \neq 0, \quad \delta\left(\frac{x}{c}\right) = c\,\delta(x). \tag{4.2.4}$$

To be allowed to take the limit $n \to \infty$ on both sides of (4.2.3) and then pass the limit below the integrals on the RHS, all limits must exist and one needs to justify exchanging limit and integrals. This is why one needs to make some restrictions on the kernel of the polynomial ensemble as its double scaling limit is not necessarily a function anymore or simply does not exist. We will make the following assumptions to guarantee existence of the limits and integrability of the different integrands involved in the computations.

Assumptions 4.2.1 *Let $n \in \mathbb{N}$. Let $\{p_j\}_{j=0,\dots,n-1}$, $\{q_j\}_{j=0,\dots,n-1}$ be bi-orthonormal families of functions composing the kernel K_n of a polynomial ensemble as defined in (2.1.24).*

1. *There exists a positive sequence $\{\nu_n\}_{n\in\mathbb{N}}$, such that for $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$, the pointwise limit*

$$K^{(\infty)}(x,y) := \lim_{n\to\infty} \frac{1}{n\nu_n} K_n\left(\frac{x}{n\nu_n}, \frac{y}{n\nu_n}\right) \tag{4.2.5}$$

   is a measurable function on $\mathbb{R}_+ \times \mathbb{R}$.
2. *There exist two real continuous functions f and g, independent of n, such that:*
   (a) *For almost all x in $\mathbb{R}_+$,*

$$\forall j \in [\![0, n-1]\!], \quad \left|\frac{1}{\nu_n} q_j\left(\frac{x}{n\nu_n}\right)\right| \leqslant \exp(f(x)), \tag{4.2.6}$$

   with $f(u) = o(u)$, as $u \to \infty$, and $u \mapsto \exp(f(u)) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ locally integrable.
   (b) *For almost all y in $\mathbb{R}$,*

$$\forall j \in [\![0, n-1]\!], \quad \left|p_j\left(\frac{y}{n\nu_n}\right)\right| \leqslant \exp(g(y)). \tag{4.2.7}$$

3. *We have, for $R > 0$,*

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_R^\infty dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \frac{1}{n} \sum_{j=0}^{n-1} \left|p_j\left(\frac{xt}{n\nu_n}\right)\right| = 0. \tag{4.2.8}$$

Remark 4.2.2 *The bounds (4.2.7) and (4.2.6) are not too restrictive in regard to the assumption on the existence of the limiting kernel (4.2.5). Indeed, if a limiting kernel exists around the origin, then it is reasonable to assume that the products $q_j(x)p_j(y)$, remains bounded independently of n on the scale $1/(n\nu_n)$ around the origin. Moreover, as we will be integrating the limiting kernel, one needs to require the bound of the functions q_j to be at least locally integrable near 0. This is given for the p_j as they are polynomials. Similarly, while (4.2.8) is only technical, it guarantees that the contribution of the polynomials to the integral in (4.2.3) remains in a bounded interval near 0, i.e. near the hard edge.*

Remark 4.2.3 *While finding a set of bi-orthonormal functions p_j, q_j is not easy in general, once found for any n, the conditions are rather easy to check. Note that imposing the conditions (4.2.7) and (4.2.6) separately on p_j and q_j constrains the choice of the bi-orthonormal system without making it unique. The reason of this choice is to decouple the arguments of the kernel. The consequence is that not all polynomial basis for the p_j are admissible anymore (see Remark 2.1.12), and one needs to find a basis where the p_j and q_j remains individually bounded; independently*

of n.

Example 4.2.4 *For Jacobi, Cauchy-Lorentz and Laguerre ensembles (2.1.33), $K^{(\infty)}$ is expressed in terms of the Bessel kernel, cf. Proposition 4.3.14. Regarding the functions p_j and q_j, one can use the choice (2.1.38), which yields, explicitly, (3.3.34) and (3.3.26). Then, echoing Remark 4.2.3, one needs to rescale by a factor ξ_n and consider instead the families $\{\xi_n^{-1}p_j\}_{j=0,\dots,n-1}$, $\{\xi_n q_j\}_{j=0,\dots,n-1}$. We take $\nu_n = 1$ and $\xi_n = 1$, for Laguerre, and $\nu_n = n$ and $\xi_n = n^{1+\alpha}$, for Jacobi and Cauchy-Lorentz (more details are given in Example 4.3.4).*

A possible choice of the polynomials p_j for those ensembles is given by (2.1.38) and one can easily find the bound, for $j = 0, \dots, n-1$,

$$\left|\frac{1}{\xi_n}p_j\left(\frac{y}{n\nu_n}\right)\right| \leqslant \exp(|y| + c), \quad c > 0. \tag{4.2.9}$$

and take $g(y) = |y| + c$; see (4.3.19).

Then, as long as the Pólya weights remains bounded on $\mathbb{R}_+$, which corresponds to $\alpha \geqslant 0$ for these ensembles, the corresponding functions q_j are simply bounded by a n-independent constant

$$\left|\frac{\xi_n}{\nu_n}q_j\left(\frac{x}{n\nu_n}\right)\right| \leqslant C, \quad C > 0, \tag{4.2.10}$$

for $j = 0, \dots, n-1$. Thus one can take $f(x) = \ln(C)$.

In the case the Pólya weight is unbounded, i.e. $\alpha < 0$, one must either select the biorthonormal functions more carefully—together with the general bound (4.3.20)—or exploit the integrable structure of the ensembles, specifically the Christoffel-Darboux formula (2.1.51) or the integral representation of the kernel (2.1.37), to verify that it is indeed uniformly bounded with respect to n.

The last assumption (4.2.8) requires more work to check and it is done in section 4.5.

Under these assumptions, the double scaling limit version of Theorem 3.3.7 around the origin, in the large n limit, yields Theorem 4.2.5. Let us stress, here, that the eigenradius and the singular values scale differently. The scale of the smallest squared eigenradius is n times larger than the scale of the smallest squared singular value. This was already observed for classical ensembles like the Laguerre ensemble (1.1.12).

Indeed, in the Laguerre case the limiting macroscopic distribution of the singular values is the quarter circle law and the mean level spacing around the origin is of order $1/n$. As for the eigenvalues, they tend to the uniform distribution in a annulus/disk. Thus, the smallest eigenvalue scales as $1/\sqrt{n}$ and so is the smallest eigenradius. Going over to squared variables yields a scaling ratio of $1/n$. A general

and simple explanation of why this scaling ratio is universal is lacking. However, from the analysis perspective, this ratio appears very naturally and is indeed universal, at least for polynomial ensembles, as shown in Theorem 4.2.5, for which a proof is given in section 4.6.

Theorem 4.2.5 *Let $k \in \mathbb{N}$ and consider a random matrix drawn from a bi-unitarily invariant ensemble on $\mathrm{GL}(n,\mathbb{C})$ having a polynomial ensemble with joint probability density (2.1.23) for the squared singular values. Under Assumptions 4.2.1, the pointwise limit*

$$f_{1,k}^{(\infty)}(r;a_1,\dots,a_k) := \lim_{n\to\infty} \frac{n}{\nu_n^{k+1}} f_{1,k}\left(\frac{r}{\nu_n};\frac{a_1}{n\nu_n},\dots,\frac{a_k}{n\nu_n}\right) \tag{4.2.11}$$

of the $1,k$-point correlation function between one squared eigenradius and k squared singular values is given by

$$\begin{aligned} f_{1,k}^{(\infty)}(r;a_1,\dots,a_k) = \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}\left(\tfrac{v}{r},t\right) \\ \times \det\begin{pmatrix} K^{(\infty)}(v,-rt) & K^{(\infty)}(v,a_c)-\delta(v-a_c) \\ K^{(\infty)}(a_b,-rt) & K^{(\infty)}(a_b,a_c) \end{pmatrix}_{b,c=1}^{k}, \end{aligned} \tag{4.2.12}$$

with δ the Dirac delta function and

$$\varphi^{(\infty)}(x,t) := x(t+x-1)e^{-x}e^{-t}. \tag{4.2.13}$$

The pointwise limit of 1-point functions, respectively on one squared singular value and one squared eigenradius, are given by

$$\rho_{SV}^{(\infty)}(a) := \lim_{n\to\infty} \tfrac{1}{\nu_n}\rho_{SV}\left(\tfrac{a}{n\nu_n}\right) = K^{(\infty)}(a,a) \tag{4.2.14}$$

and

$$\rho_{EV}^{(\infty)}(r) := \lim_{n\to\infty} \tfrac{n}{\nu_n}\rho_{EV}\left(\tfrac{r}{\nu_n}\right) = \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}\left(\tfrac{v}{r},t\right) K^{(\infty)}(v,-rt)\,. \tag{4.2.15}$$

Remark 4.2.6 *Rewriting the result as follows*

$$\frac{1}{\nu_n(n\nu_n)^k} f_{1,k}\left(\frac{r}{\nu_n};\frac{a_1}{n\nu_n},\dots,\frac{a_k}{n\nu_n}\right) \underset{n\to\infty}{=} \frac{1}{n^{1+k}}\left[f_{1,k}^{(\infty)}(r;a_1,\dots,a_k)+o(1)\right], \tag{4.2.16}$$

one can see that the scaling limit of the $1,k$-point function at the origin is of order $O(1/n^{1+k})$ as $n\to\infty$. We stress that (4.2.16) is on the level of probability densities, which are thus properly normalized. It should be noted, however, that $\rho_{EV}^{(\infty)}, \rho_{SV}^{(\infty)}$, and $f_{1,k}^{(\infty)}$ are not probability densities.

The double scaling limit of the $1,k$-cross-covariance density follows directly from

the definition (2.3.28) and is given by the following corollary.

Corollary 4.2.7 *With the same assumptions as in Theorem 4.2.5, the pointwise limit*

$$\mathrm{cov}_{1,k}^{(\infty)}(r;a_1,\dots,a_k) := \lim_{n\to\infty} \frac{n}{\nu_n^{k+1}} \mathrm{cov}_{1,k}\left(\frac{r}{\nu_n};\frac{a_1}{n\nu_n},\dots,\frac{a_k}{n\nu_n}\right) \tag{4.2.17}$$

of the $1,k$-cross-covariance density functions between one squared eigenradius and k squared singular values is given by

$$\mathrm{cov}_{1,k}^{(\infty)}(r;a_1,\dots,a_k) = \partial_\mu \det\left[K^{(\infty)}(a_b,a_c) + \mu\, \mathbf{C}^{(\infty)}(r;a_b,a_c)\right]_{b,c=1}^{k}\Big|_{\mu=0} \tag{4.2.18}$$

with

$$\mathbf{C}^{(\infty)}(r;a_b,a_c) := \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}\left(\tfrac{v}{r},t\right) K^{(\infty)}\left(a_b,-rt\right)\left[\delta(v-a_c) - K^{(\infty)}(v,a_c)\right], \tag{4.2.19}$$

where δ is the Dirac delta function. In particular for $k=1$, the limiting cross-covariance reads

$$\mathrm{cov}^{(\infty)}(r;a) = \mathbf{C}^{(\infty)}(r;a,a). \tag{4.2.20}$$

Proof. Using Theorem 4.2.5 together with the definition (2.3.28), and noting that each of the functions $f_{1,k}$, $f_{1,0}=\rho_{\mathrm{EV}}$, and $f_{0,k}$ admits an explicit double scaling limit, we obtain the double scaling limit of the $1,k$-cross-covariance density as

$$\begin{aligned}\mathrm{cov}_{1,k}^{(\infty)}(r;a_1,\dots,a_k) &= \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}\left(\tfrac{v}{r},t\right) \\ &\quad \times \det\begin{pmatrix} 0 & K^{(\infty)}(v,a_c) - \delta(v-a_c) \\ K^{(\infty)}(a_b,-rt) & K^{(\infty)}(a_b,a_c)\end{pmatrix}_{b,c=1}^{k} \\ &= \sum_{l,m=1}^{k} (-1)^{l+m} \det\left[K^{(\infty)}(a_b,a_c)\right]_{\substack{b\in[\![1,k]\!]\setminus\{l\}\\ c\in[\![1,k]\!]\setminus\{m\}}} \mathbf{C}^{(\infty)}(r;a_l,a_m).\end{aligned} \tag{4.2.21}$$

Applying Jacobi's formula for derivative of determinants (A.2.4) yields the result. ∎

4.3 Pólya ensembles

For the subclass of Pólya ensembles, the formulas of Theorem 4.2.5 can be made more explicit by computing the double integral. Furthermore, as Pólya ensembles enjoy more structure and are entirely defined by their weight function, one can conveniently translate and make precise Assumptions 4.2.1 directly on the Pólya weight.

Thus, in the case of Pólya ensembles, the following assumptions (cf. [89, Assump-

tions 2.2]) are sufficient for the existence of a hard edge at the origin and guarantee the existence of the limiting kernel (4.2.5). The final assumption is an additional condition not present in [89, Assumptions 2.2], whereas the remaining assumptions have been adapted to accommodate the new scalings ξ_n and ν_n.

Assumptions 4.3.1 *Let $n \in \mathbb{N}$. We assume that the analytic continuation of the Mellin transform $\mathcal{M}w_n$ is holomorphic on $\mathbb{C}\setminus(]-\infty,1[\cup]n,\infty[)$ and that there exist two real positive sequences, $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$, such that:*

1. *For all fixed $s \in \mathbb{C}\setminus]-\infty,1]$, the pointwise limit*

$$\lim_{n\to\infty} \xi_n \nu_n^{s-1} \mathcal{M}w_n(s) := \tilde{w}^{(\infty)}(s) \tag{4.3.1}$$

   exist and is a measurable function.
2. *There exists a constant $C > 0$, n-independent, such that*

$$\frac{1}{\xi_n \nu_n^{s-1} |\mathcal{M}w_n(s)|} \leqslant C, \quad s \in [1,n]. \tag{4.3.2}$$

3. *We have the domination, for all $z \in \mathbb{C}\setminus(]-\infty,1[\cup]n,\infty[)$,*

$$\sup_{\arg(z)=\vartheta} \{|\xi_n \nu_n^z \mathcal{M}w_n(1+z)|\} \leqslant D(\vartheta), \quad \forall n \in \mathbb{N}, \tag{4.3.3}$$

   where $0 < D(\vartheta) < \infty$, for all $\vartheta \in [\pi/2, \pi[$.
4. *There exists $j_* \in [\![0, n-1]\!]$, and a positive function h, both n-independent, such that*

$$\frac{1}{\xi_n \nu_n^j |\mathcal{M}w_n(j+1)|} \leqslant \frac{1}{h(j)} \leqslant C, \quad j \in [j_*, n-1], \tag{4.3.4}$$

   and the function h has the root asymptotic

$$\limsup_{j\to\infty} \frac{1}{h(j)^{1/j}} = 0. \tag{4.3.5}$$

Remark 4.3.2 *The scaling ξ_n does not explicitly play a role in the further computation. To see this one can look at (4.3.24), Proposition 4.3.10 and notice that only ratios $\mathcal{M}w_n(x)/\mathcal{M}w_n(y)$ are involved. This is reminiscent of the fact the choice of bi-orthonormal functions composing the kernel is not uniquely given; cf. Remark 2.1.12.*

While we only assume existence of the pointwise limit of the scaled Mellin transform of the weight, $\tilde{w}^{(\infty)}$, one also needs existence of the pointwise limit of the weight itself for the double scaling limit of ρ_{EV} (2.1.47). The existence of this limit is actually implied by Assumptions 4.3.1. Hence the following Lemma.

Lemma 4.3.3 *Under Assumptions 4.3.1, the pointwise limit*

$$\lim_{n\to\infty} \frac{\xi_n}{\nu_n} w_n\left(\frac{x}{\nu_n}\right) = \mathcal{M}^{-1}\tilde{w}^{(\infty)}(x) =: w^{(\infty)}(x) \tag{4.3.6}$$

holds for almost all $x \in \mathbb{R}_+$ and we have $\tilde{w}^{(\infty)}(s) = \mathcal{M}w^{(\infty)}(s)$ for fixed $s \in \mathbb{C}\backslash]-\infty, 1]$.

Proof. Let us consider the following contour $\mathcal{C}_1(\theta) = \left\{1 + ie^{\mathrm{sgn}(\tau)i\theta}\tau,\ \tau \in \mathbb{R}\right\}$ and $\theta \in]0, \frac{\pi}{2}[$, which plays the role of a regularisation function for the inverse Mellin transform $\mathcal{M}^{-1}$, ensuring the convergence of the integral independently of $w^{(\infty)}$ [89, Proof of Lemma 2.5]. Due to (4.3.3), we explicitly have

$$\begin{aligned} \left|\mathcal{M}^{-1}\tilde{w}^{(\infty)}(x)\right| &= \left|\int_{\mathcal{C}_1(\theta)} \frac{ds}{2\pi i} x^{-s} \lim_{n\to\infty} \xi_n \nu_n^{s-1} \mathcal{M}w_n(s)\right| \\ &\leqslant D(\pi/2+\theta)\left|\int_{\mathcal{C}_1(\theta)} \frac{ds}{2\pi i} x^{-s}\right| < \infty. \end{aligned} \tag{4.3.7}$$

By dominated convergence theorem, the limit $n \to \infty$ can be pulled out of the integral and this yields the claim

$$\begin{aligned} \mathcal{M}^{-1}\tilde{w}^{(\infty)}(x) &= \int_{\mathcal{C}_1(\theta)} \frac{ds}{2\pi i} x^{-s} \lim_{n\to\infty} \xi_n \nu_n^{s-1} \mathcal{M}w_n(s) \\ &= \lim_{n\to\infty} \frac{\xi_n}{\nu_n} \int_{\mathcal{C}_1(\theta)} \frac{ds}{2\pi i} \left(\frac{x}{\nu_n}\right)^{-s} \mathcal{M}w_n(s) = \lim_{n\to\infty} \frac{\xi_n}{\nu_n} w_n\left(\frac{x}{\nu_n}\right), \end{aligned} \tag{4.3.8}$$

which holds only for almost all $x \in \mathbb{R}_+$, by Mellin inversion theorem. Applying again the Mellin transform on both sides of the above equation finishes the proof. ■

Example 4.3.4 *For Laguerre, Jacobi and Cauchy-Lorentz ensembles, the respective Pólya weights are given by (2.1.33) where α, β are fixed, i.e. independent of n. The Laguerre weight being independent of n, there is no need to rescale, thus $\nu_n = 1$ and $\xi_n = 1$. For Jacobi and Cauchy-Lorentz weights, taking $\nu_n = n$ and $\xi_n = n^{1+\alpha}$ yields*

$$w^{(\infty)}_{\mathrm{Lag}}(x) = w^{(\infty)}_{\mathrm{Jac}}(x) = w^{(\infty)}_{\mathrm{CL}}(x) = x^\alpha e^{-x}. \tag{4.3.9}$$

Starting from the Mellin transform of the respective weights

$$\begin{aligned} &\mathcal{M}w_{\mathrm{Lag}}(s) = \Gamma(s+\alpha), \quad \mathcal{M}w_{\mathrm{Jac}}(s) = \mathrm{B}(s+\alpha, \beta+n), \\ &\mathcal{M}w_{\mathrm{CL}}(s) = \mathrm{B}(s+\alpha, n-1-s+\beta-\alpha), \end{aligned} \tag{4.3.10}$$

it is easy to check that we have, as expected,

$$\tilde{w}^{(\infty)}_{\mathrm{Lag}}(s) = \tilde{w}^{(\infty)}_{\mathrm{Jac}}(s) = \tilde{w}^{(\infty)}_{\mathrm{CL}}(s) = \Gamma(s+\alpha). \tag{4.3.11}$$

Regarding the n-independent bound h (4.3.4), for Laguerre, Jacobi and Cauchy-

Lorentz ensembles we have, respectively,

$$\begin{aligned}
&\xi_n \nu_n^j \mathcal{M} w_{\mathrm{Lag}}(j+1) = \Gamma(j+1+\alpha), \quad \alpha > -1, \\
&\xi_n \nu_n^j \mathcal{M} w_{\mathrm{Jac}}(j+1) = n^{j+1+\alpha} \mathrm{B}(j+1+\alpha, \beta+n), \quad \alpha > -1,\ \beta > 0, \\
&\xi_n \nu_n^j \mathcal{M} w_{\mathrm{CL}}(j+1) = n^{j+1+\alpha} \mathrm{B}(j+1+\alpha, n-j-2+\beta-\alpha), \quad \alpha > -1,\ \beta-\alpha > 1.
\end{aligned} \tag{4.3.12}$$

Using the bound[3]*, for* $j \in [1, n-1]$*,*

$$j^{j+1+\alpha} \mathrm{B}(j+1+\alpha, \beta+j+1) \leqslant n^{j+1+\alpha} \mathrm{B}(j+1+\alpha, \beta+n) \tag{4.3.13}$$

and the bound[4]*, for* $j \in [\max\{2(\beta-2)-\alpha, 0\}, n-1]$*,*

$$\Gamma(j+1+\alpha) \leqslant n^{j+1+\alpha} \mathrm{B}(j+1+\alpha, n-j-2+\beta-\alpha), \tag{4.3.14}$$

one can choose, respectively, the function h *and* j_**, to be*

$$\begin{aligned}
&h(j) = h_{\mathrm{Lag}}(j) := \Gamma(j+1+\alpha), \quad j_* = \max\{1, \lceil 1-\alpha \rceil\}, \\
&h(j) = h_{\mathrm{Jac}}(j) := j^{j+1+\alpha} \mathrm{B}(j+1+\alpha, \beta+j+1), \quad j_* = \max\{1, \lceil 1-\alpha \rceil, \lceil 1+\beta \rceil\}, \\
&h(j) = h_{\mathrm{CL}}(j) := \Gamma(j+1+\alpha), \quad j_* = \max\{1, \lceil 1-\alpha \rceil, \lceil 2(\beta-2)-\alpha \rceil\},
\end{aligned} \tag{4.3.15}$$

where $\lceil \cdot \rceil$ *is the ceiling function. In each case,* h *has the root asymptotic* (4.3.5)*. Furthermore, with those particular choices*[5] *of* j_**, the functions* $h_{\mathrm{Lag}}, h_{\mathrm{Jac}}, h_{\mathrm{CL}}$ *are strictly increasing on* $[j_*, n-1]$ *and bounded from below by* 1*.*

Remark 4.3.5 *It is interesting to notice that the set of Pólya ensembles verifying Assumptions 4.3.1 enjoys the same closure property as the set of all Pólya ensembles; cf.* (2.1.31)*. Indeed, given two complex square matrices* X *and* Y *drawn from different (or same) Pólya ensembles whose respective Pólya weight* w_X *and* w_Y *verify Assumptions 4.3.1, then, the matrix* XY *also follows a Pólya ensemble, whose corresponding Pólya weight is given by the multiplicative convolution*

$$w_{XY}(x) := (w_X * w_Y)(x) = \int_0^\infty w_X\left(\frac{x}{y}\right) w_Y(y) \frac{dy}{y}. \tag{4.3.16}$$

Due to the properties of the Mellin transform (2.3.4)*, we have*

$$\mathcal{M} w_{XY}(s) = \mathcal{M} w_X(s)\, \mathcal{M} w_Y(s), \tag{4.3.17}$$

and it is then easy to see that the Pólya weight w_{XY} *also verifies Assumptions 4.3.1 with the scalings*

$$\nu_{n,XY} = \nu_{n,X}\, \nu_{n,Y}, \quad \xi_{n,XY} = \xi_{n,X}\, \xi_{n,Y}, \tag{4.3.18}$$

[3] This bound can be checked by looking at the ratio LFH/RHS, locating the maximum and verifying it is indeed less than one for this range of j. One can also use Stirling related bounds in the process; cf. (4.5.21).

[4] This bound is immediate when writing the Beta function as a ratio of Gamma functions.

[5] We do not claim these choices of j_* to be optimal. Other choices are possible.

where $\nu_{n,X}, \xi_{n,X}$ (resp. $\nu_{n,Y}, \xi_{n,Y}$) are the scalings associated with the Pólya ensemble of X (resp. Y).

Comparing Assumptions 4.2.1 and Assumptions 4.3.1, one has the following lemma.

Lemma 4.3.6 (Consistency of assumptions)
1. *The first assumption* (4.3.1) *implies* (4.2.5).
2. *The second assumption* (4.3.2) *implies* (4.2.7).
3. *The last assumption, given by* (4.3.4) *and* (4.3.5), *implies* (4.2.8).

Proof. The first assumption (4.3.1) implies (4.2.5) through Proposition 4.3.10.

To show that the second assumption (4.3.2) implies (4.2.7), one can use the explicit expression of the polynomials p_j (2.1.38), and the binomial identity, to get the following bound, for $j = 0, \dots, n-1$,

$$\left|\frac{1}{\xi_n} p_j\left(\frac{y}{n\nu_n}\right)\right| = \left|\sum_{c=0}^{j} \binom{j}{c} \frac{1}{\xi_n \nu_n^c \mathcal{M}w_n(c+1)} \left(\frac{-y}{n}\right)^c\right| \leqslant C\left(1+\frac{|y|}{n}\right)^j \leqslant C\exp(|y|), \tag{4.3.19}$$

with C some n-independent positive constant.

The last claim requires more work to prove and is given by Lemma 4.5.1, for which the proof is given in section 4.5. ■

As can be seen, Lemma 4.3.6 does not establish that the assumptions made for the subclass of Pólya ensembles fully imply those required for polynomial ensembles. In particular, the implication of (4.2.6) by Assumptions 4.2.1 is lacking.

Note that the assumption (4.3.3) is likely to imply the asymptotic behaviour $f(u) = o(u)$, as $u \to \infty$, in (4.2.6), yet it is neither clear nor shown. Regarding the n-independent and locally integrable bound, it is given by [89, Corollary 2.6] that we reproduce here in a slightly generalized version—accounting for the different assumptions and definitions of the functions; cf. Remark 4.3.11. The original proposition only considers $j = n$.

Proposition 4.3.7 ([89, Corollary 2.6 modified]) *Under Assumptions 4.3.1, the scaled functions q_j given by* (2.1.40) *can be bounded on any bounded interval $]a, b] \subset \mathbb{R}_+$ as*

$$\left|\frac{\xi_n}{\nu_n} q_j\left(\frac{x}{(j+1)\nu_n}\right)\right| \leqslant \frac{c}{x(\ln(x)^2+1)}, \quad j \in [\![0, n]\!] \tag{4.3.20}$$

with c an n-independent constant depending only on the half-open interval $]a, b]$, so that this inequality also holds when taking the limit $n \to \infty$ on both sides.

Proof. The proof is essentially the same as the original proof given in [89], accounting for the scalings ξ_n, ν_n. The notable difference being that the proof only considers

the case $j = n$. We stress here that the proof can be adapted for $j \in [\![0, n]\!]$ by rewriting the scaled q_j in the form

$$\frac{\xi_n}{\nu_n} q_j \left(\frac{x}{(j+1)\nu_n} \right) = \int_{\mathcal{C}_1(\theta)} \frac{ds}{2\pi i} x^{-s} \frac{\Gamma(j+1-s)(j+1)^s}{\Gamma(j+1)\Gamma(1-s)} \xi_n \nu_n^{s-1} \mathcal{M} w_n(s), \tag{4.3.21}$$

where $\mathcal{C}_1(\theta) = \{1 + ie^{\mathrm{sgn}(\tau) i\theta}\tau,\, \tau \in \mathbb{R}\}$. The term $|\xi_n \nu_n^{s-1} \mathcal{M} w_n(s)|$ is bounded uniformly in n on $\mathcal{C}_1(\theta)$ due to (4.3.3), thus the same analysis follows where the original n-dependence of the ratio of gamma functions on the RHS of (4.3.21) has been replaced by j. ■

Let us mention that the last of the Assumptions 4.2.1, given by (4.3.4) and (4.3.5), could possibly be lifted due to some property of the Pólya frequency functions such as their log-concavity [60, 65, 130].

All that being said, due to assumption (4.3.3), it remains unclear whether the set of Pólya ensembles satisfying Assumptions 4.3.1 is strictly contained within the set of polynomial ensembles satisfying Assumptions 4.2.1. Notwithstanding, in both cases, the formulas (4.2.12), (4.2.14) and (4.2.15) hold, given that the limiting kernel does not grow too fast in the first argument. Hence the following theorem, for which a proof is given in subsection 4.6.2.

Theorem 4.3.8 *Let $k \in \mathbb{N}$ and consider a random matrix drawn from a bi-unitarily invariant ensemble on $\mathrm{GL}(n, \mathbb{C})$ having a Pólya ensemble with joint probability density (2.1.23) for the squared singular values. Under Assumptions 4.3.1, instead of Assumptions 4.2.1 and assuming that $x \mapsto K^{(\infty)}(x, y)$ grows at most sub-exponentially as $x \to \infty$, uniformly in $y \in \mathbb{R}$, the results of Theorem 4.2.5 hold.*

Remark 4.3.9 *We strongly believe that the extra assumption on the growth of the limiting kernel in the first argument is actually always verified. The first reason is that it is likely to be implied by Assumptions 4.2.1. The second reason is heuristic. The limiting kernel encodes the local correlations of the points—particles—and gives the limiting level density (see (4.2.14)). Thus, if there is a non trivial limiting kernel around the hard edge, it should remain bounded in the direction of the bulk; cf. Figure 2.2.*

While the formulas in Theorem 4.2.5 seem to be quite general and are expressed explicitly in terms of the limiting kernel, they involve a double non-compact integral, which might not be easily computed. Yet, the structure of Pólya ensembles enables to go further by computing this double integral.

Let us first recall that, for Pólya ensembles, the kernel admits an integral representation (2.1.37), with the bi-orthonormal sets of functions $\{q_j\}_{j=0}^{n-1}$ and $\{p_j\}_{j=0}^{n-1}$ given by (2.1.38). We assume here the n-times differentiability of w_n, i.e., $w_n \in \mathrm{C}^n(\mathbb{R})$, to have a well-defined function q_n. If the weight is only $(n-1)$-differentiable one needs to modify the formula to (2.1.39).

One can rescale q_n and p_{n-1} as follows, where q_n is expressed with the help of an inverse Mellin transform (the regularization function for the complex integral is omitted here; see (2.1.40)),

$$Q_n(x) := \frac{\xi_n}{\nu_n} q_n\left(\frac{x}{n\nu_n}\right) = \int_{\mathcal{C}_1} \frac{ds}{2i\pi} \frac{\Gamma(n+1-s)n^{s-1}}{\Gamma(n)\Gamma(1-s)} \xi_n \nu_n^{s-1} \mathcal{M}w_n(s) x^{-s}, \tag{4.3.22}$$

with $\mathcal{C}_1 := 1 + i\mathbb{R}$, and

$$P_{n-1}(y) := \frac{1}{\xi_n} p_{n-1}\left(\frac{y}{n\nu_n}\right) = \sum_{c=0}^{n-1} \binom{n-1}{c} \frac{(-y)^c}{n^c \xi_n \nu_n^c \mathcal{M}w_n(c+1)}. \tag{4.3.23}$$

The scaled version of the kernel (2.1.37) is then

$$\frac{1}{n\nu_n} K_n\left(\frac{x}{n\nu_n}, \frac{y}{n\nu_n}\right) = \int_0^1 dt\, Q_n(xt) P_{n-1}(yt), \tag{4.3.24}$$

for which the scaling limit is given by the following proposition, which is a modified version of [89, Prop.2.8], with the corresponding Assumptions 4.3.1 accounting for the additional scaling ν_n. We relax the constraint on the domain of y as the proposition is true for any $y \in \mathbb{R}$.

Proposition 4.3.10 ([89, Prop.2.8 modified]) *The hard edge limit of the kernel (2.1.37) for Pólya ensembles satisfying the Assumptions 4.3.1 is given by the pointwise limit*

$$K^{(\infty)}(x,y) := \lim_{n\to\infty} \frac{1}{n\nu_n} K_n\left(\frac{x}{n\nu_n}, \frac{y}{n\nu_n}\right) = \int_0^1 dt\, Q^{(\infty)}(xt) P^{(\infty)}(yt), \tag{4.3.25}$$

for any fixed $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$, where,

$$P^{(\infty)}(y) := \lim_{n\to\infty} P_{n-1}(y) = \sum_{j=0}^{\infty} \frac{(-y)^j}{j!\, \tilde{w}^{(\infty)}(j+1)}. \tag{4.3.26}$$

$$Q^{(\infty)}(x) := \lim_{n\to\infty} Q_n(x) = \int_{\mathcal{C}_1(\theta)} \frac{ds}{2i\pi} \frac{\tilde{w}^{(\infty)}(s)}{\Gamma(1-s)} x^{-s} \tag{4.3.27}$$

with $\mathcal{C}_1(\theta) = \left\{1 + ie^{\mathrm{sgn}(\tau) i\theta}\tau,\ \tau \in \mathbb{R}\right\}$ and $\theta \in]0, \frac{\pi}{2}[$. sgn is the signum function.

Remark 4.3.11 *The definition of the contour $\mathcal{C}_1(\theta)$ plays the role of a regularisation function, ensuring the convergence of the integral independently of $w^{(\infty)}$ [89, Proof of Lemma 2.5]. Note that while the definition of Q_n in [89, Prop.2.8] do not include the factor $\Gamma(n+1-s)n^{s-1}/\Gamma(n)$, they share the same scaling limit $Q^{(\infty)}$, by the same arguments used in the proof of [89, Prop.2.8]. Up to the factor just mentioned, Q_n and P_{n-1} correspond, respectively, to the functions $\tilde{J}\omega^{(n)}$ and $J\omega^{(n)}$ in [89].*

It is known that for classical Pólya ensembles like Laguerre and Jacobi ensembles, the limiting kernel at the hard edge at the origin can be expressed in terms of the Bessel kernel (2.2.21), which admits the following integral representation (cf. [143]), for which a proof is given in section A.1.

Proposition 4.3.12 *For all $x, y \in \mathbb{R}$,*

$$K_\alpha^{\mathrm{Bes}}(x,y) := \frac{1}{4}\int_0^1 dt J_\alpha(\sqrt{xt})J_\alpha(\sqrt{yt}) = \frac{\sqrt{y}J'_\alpha(\sqrt{y})J_\alpha(\sqrt{x}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}, \tag{4.3.28}$$

with J_α the usual Bessel function of the first kind of order α.

Remark 4.3.13 *In the formula (4.2.12), the second argument of the limiting kernel is negative. This is not an issue. Albeit J_α is analytic on $\mathbb{C}\backslash]-\infty, 0]$, for negative arguments, one can use the relation with the modified Bessel function I_α*

$$\forall x \in \mathbb{R}_+, \quad J_\alpha(ix) = i^\alpha I_\alpha(x), \quad J'_\alpha(ix) = i^{\alpha-1}I'_\alpha(x). \tag{4.3.29}$$

For instance for $x, y \in \mathbb{R}_+$,

$$K_\alpha^{\mathrm{Bes}}(x,-y) = i^\alpha \frac{\sqrt{y}I'_\alpha(\sqrt{y})J_\alpha(\sqrt{x}) - \sqrt{x}J'_\alpha(\sqrt{x})I_\alpha(\sqrt{y})}{2(x+y)} \tag{4.3.30}$$

and

$$K_\alpha^{\mathrm{Bes}}(-x,-y) = (-1)^\alpha \frac{\sqrt{y}I'_\alpha(\sqrt{y})I_\alpha(\sqrt{x}) - \sqrt{x}I'_\alpha(\sqrt{x})I_\alpha(\sqrt{y})}{2(y-x)}. \tag{4.3.31}$$

In general, the Bessel kernel appears whenever the scaled Pólya weight converges to the Laguerre weight in the double scaling limit.

Proposition 4.3.14 *If $\tilde{w}^{(\infty)}(s) = \Gamma(s+\alpha)$, with $\mathrm{Re}(\alpha) > -1$, then,*

$$K^{(\infty)}(x,y) = \left(\frac{x}{y}\right)^{\frac{\alpha}{2}} \int_0^1 dt J_\alpha(2\sqrt{xt})J_\alpha(2\sqrt{yt}) = 4\left(\frac{x}{y}\right)^{\frac{\alpha}{2}} K_\alpha^{\mathrm{Bes}}(4x,4y) \tag{4.3.32}$$

with $K^{(\infty)}$ defined as in Proposition 4.3.10.

Proof of Proposition 4.3.14. Using the definition of $P^{(\infty)}$ and $Q^{(\infty)}$ given in Proposition 4.3.10 we get

$$P^{(\infty)}(y) = \sum_{j=0}^{\infty} \frac{(-y)^j}{j!\,\Gamma(j+1+\alpha)} = y^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{y}) \tag{4.3.33}$$

and

$$Q^{(\infty)}(x) = \int_{\mathcal{C}_1(\theta)} \frac{ds}{2i\pi} \frac{\Gamma(s+\alpha)}{\Gamma(1-s)} x^{-s}. \tag{4.3.34}$$

In this case the integral converges even for $\theta = 0$, yielding $\mathcal{C}_1(0) = 1 + i\mathbb{R} = \mathcal{C}_1$. Thus, we get

$$Q^{(\infty)}(x) = \int_{\mathcal{C}_1} \frac{ds}{2i\pi} \frac{\Gamma(s+\alpha)}{\Gamma(1-s)} x^{-s} = x^{\frac{\alpha}{2}} J_\alpha(2\sqrt{x}). \tag{4.3.35}$$

By Proposition 4.3.12 the result follows. ■

Using Proposition 4.3.10, the scaling limit of the $1, k$-cross-covariance density function, is then given by the following corollary, which can be seen as the double scaling limit version of Corollary 3.3.9.

Corollary 4.3.15 *For Pólya ensembles verifying the assumptions Assumptions 4.3.1, the scaling limit of the $1, k$-cross-covariance density functions between one squared eigenradius and k squared singular values is given by (4.2.18) where $\mathbf{C}^{(\infty)}$ can be recast into the form*

$$\begin{aligned} \mathbf{C}^{(\infty)}(r; a_1, a_2) &= \sum_{\gamma=0,1} H_\gamma^{(\infty)}(r, a_1) \left[r^{-1} e^{-\frac{a_2}{r}} \left(\frac{a_2}{r} - 1 \right)^{1-\gamma} - V_\gamma^{(\infty)}(r, a_2) \right] \\ &= \partial_r \left[H_0^{(\infty)}(r, a_1) \left(e^{-\frac{a_2}{r}} - r V_1^{(\infty)}(r, a_2) \right) \right], \end{aligned} \tag{4.3.36}$$

for almost all $(r, a_1, a_2) \in \mathbb{R}_+^3$, where, for $\gamma = 0, 1$,

$$H_\gamma^{(\infty)}(x, y) := \int_0^1 du\; Q^{(\infty)}(yu) \partial_u^\gamma \left[u^\gamma \frac{\rho_{\mathrm{EV}}^{(\infty)}(xu)}{w^{(\infty)}(xu)} \right], \tag{4.3.37}$$

$$V_\gamma^{(\infty)}(x, y) := \int_0^1 du\; P^{(\infty)}(yu) \left(u \partial_u \right)^{1-\gamma} w^{(\infty)}(xu), \tag{4.3.38}$$

with $P^{(\infty)}$ and $Q^{(\infty)}$ given respectively by (4.3.26) and (4.3.27), and

$$\rho_{EV}^{(\infty)}(x) = w^{(\infty)}(x) \sum_{k=0}^{\infty} \frac{x^k}{\tilde{w}^{(\infty)}(k+1)}. \tag{4.3.39}$$

Proof of Corollary 4.3.15. Let us first show that the scaling limit $\rho_{EV}^{(\infty)}$ is indeed given by (4.3.39). From [91, Eq. (4.7)] the scaling version of the 1-point function on the squared eigenradii is given by

$$\tfrac{n}{\nu_n} \rho_{EV}\left(\tfrac{x}{\nu_n} \right) = \frac{\xi_n}{\nu_n} w_n \left(\frac{x}{\nu_n} \right) \sum_{j=0}^{n-1} \frac{x^j}{\xi_n \nu_n^j \mathcal{M} w_n(j+1)}. \tag{4.3.40}$$

Using the bound assumption (4.3.4), one gets the following bound

$$\sum_{j=0}^{n-1} \frac{x^j}{\xi_n \nu_n^j \mathcal{M} w_n(j+1)} \leqslant \sum_{j=0}^{j_*-1} \frac{x^j}{\xi_n \nu_n^j \mathcal{M} w_n(j+1)} + \sum_{j=j_*}^{n-1} \frac{x^j}{h(j)}, \tag{4.3.41}$$

where we remind that j_* is n-independent. By Cauchy-Hadamard theorem, due to the root asymptotic (4.3.5), the second sum on the RHS is a partial sum of a convergent series with infinite radius of convergence. The first j_* summands are bounded by a constant independent of n by assumption (4.3.2). Moreover, as h is also n-independent, this provides an upper bound to apply dominated convergence theorem. Thus, one is allowed to take the limit $n \to \infty$ inside the sum on the LHS and using the limits (4.3.1) and (4.3.6)—which holds almost everywhere on $\mathbb{R}_+$—yields the claim.

For the remaining part of the claim, we start from Corollary 3.3.9 and apply Jacobi's formula for the derivative of the determinant (A.2.4), to get

$$\mathrm{cov}_{1,k}(r; a_1, \dots, a_k) = \frac{(n-k)!}{(n-1)!} \sum_{l,m=1}^{k} (-1)^{l+m} \mathbf{C}_n(r; a_l, a_m) \det[K_n(a_b, a_c)]_{\substack{b \in [\![1,k]\!] \setminus \{l\} \\ c \in [\![1,k]\!] \setminus \{m\}}}. \tag{4.3.42}$$

One then proceeds with the appropriate scalings. The integrands in (3.3.21) and (3.3.20) are bounded by n-independent integrable functions on $]0,1]$, as it can be seen with (4.3.41), (4.3.19) and with the bound given in [89, Corollary 2.6], see (4.3.20). Moreover, $\tilde{w}^{(\infty)}$ is a measurable function by assumption (4.3.1). By Lebesgue's dominated convergence theorem, the limit $n \to \infty$ can be taken inside the integrals. Using, Proposition 4.3.10, all limits exist and one arrives at the first expression in (4.3.36).

Then, using the relations

$$\begin{aligned} H_1^{(\infty)}(x,y) &= \partial_x \left[x H_0^{(\infty)}(x,y) \right], \\ V_0^{(\infty)}(x,y) &= x \partial_x V_1^{(\infty)}(x,y), \end{aligned} \tag{4.3.43}$$

which are the scaled versions of the relation given in [10, Sec. 4.1], finishes the proof. ∎

Example 4.3.16 *When* $\tilde{w}^{(\infty)}(s) = \Gamma(s+\alpha)$ *(cf. Example 4.3.4), with* $\mathrm{Re}(\alpha) > -1$, $P^{(\infty)}$ *and* $Q^{(\infty)}$ *reduce, respectively, to* (4.3.33) *and* (4.3.35) *and*

$$\rho_{EV}^{(\infty)}(x) = \frac{\gamma(\alpha, x)}{\Gamma(\alpha)}, \tag{4.3.44}$$

$$H_0^{(\infty)}(x,y) = x^{-\alpha} y^{\alpha/2} \int_0^1 du\, J_\alpha(2\sqrt{yu})\, u^{-\alpha/2} e^{xu} \frac{\gamma(\alpha, xu)}{\Gamma(\alpha)}, \tag{4.3.45}$$

$$V_1^{(\infty)}(x,y) = x^{\alpha} y^{-\alpha/2} \int_0^1 du\, J_\alpha(2\sqrt{yu})\, u^{\alpha/2} e^{-xu}, \tag{4.3.46}$$

where $(\alpha, x) \mapsto \gamma(\alpha, x)$ *is the lower incomplete gamma function. Note that there is no issue at* $\alpha = 0$ *as the function* $(\alpha, z) \mapsto \frac{\gamma(\alpha,z)}{\Gamma(\alpha)}$ *is holomorphic on* $\mathbb{C}^2$.

4.4 Example: Muttalib-Borodin ensembles

The Laguerre type Muttalib-Borodin ensemble with parameter $\theta' > 0$ and weight function $x^{\alpha'}e^{-x}$, $\alpha' > -1$ is the following probability density function on $\mathbb{R}_+^n$ [23, 118]

$$\frac{1}{Z_n}\Delta_n(x)\Delta_n(x^{\theta'})\det(x)^{\alpha'}e^{-\operatorname{Tr}(x)}. \tag{4.4.1}$$

This type of ensemble was introduced in the physics literature by Muttalib [118] to describe disordered conductors in the metallic regime and to account for specific two-body interactions in the model's effective Hamiltonian. It has then been studied by Borodin [23] who focused on the three classical types of weight: Laguerre, Jacobi, Hermite.

We will consider, here, only the Laguerre type (4.4.1), on $\mathbb{R}_+^n$. Under the change of variables

$$x \mapsto x^{1/\theta'}, \quad \theta = \frac{1}{\theta'}, \quad \alpha = \frac{\alpha'+1}{\theta'} - 1, \tag{4.4.2}$$

it becomes a Pólya ensemble with weight $w_{\mathrm{MB}}(x) = x^{\alpha}e^{-x^{\theta}}$, where $\theta > 0$ and $\alpha > -1$. As the weight is independent of n, it remains the same in the large n limit and thus $\nu_n = \xi_n = 1$. Its Mellin transform is given by

$$\mathcal{M}w_{\mathrm{MB}}(s) = \tilde{w}_{\mathrm{MB}}^{(\infty)}(s) = \frac{1}{\theta}\Gamma\left(\frac{\alpha+s}{\theta}\right). \tag{4.4.3}$$

Therefore, from (2.1.38), the functions p_{n-1} and q_n composing the kernel K_n (2.1.37) are given by

$$p_{n-1}(y) = \theta\sum_{c=0}^{n-1}\binom{n-1}{c}\frac{(-y)^c}{\Gamma\left(\frac{\alpha+c+1}{\theta}\right)} \tag{4.4.4}$$

and

$$q_n(x) = \frac{w_{\mathrm{MB}}(x)}{n!}\sum_{c=0}^{n}\frac{(x^{\theta})^c}{c!}\sum_{l=0}^{c}\binom{c}{l}(-1)^l\frac{\Gamma(\theta l+\alpha+1+n)}{\Gamma(\theta l+\alpha+1)}. \tag{4.4.5}$$

By properties of the Γ function, one can easily check that first assumptions in Assumptions 4.3.1 are verified. Indeed, Γ being holomorphic on $\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$, it is holomorphic on $\mathbb{C}\setminus(]-\infty,1[\cup]n,\infty[)$. The very crude bound $1/2 < \Gamma(x)$, for $x > 0$, verifies (4.3.2) and the domination (4.3.3) is verified by using a Stirling like bound, such as [89, Eq.(33)].

For the last assumption, as $\nu_n = \xi_n = 1$, one can take

$$h(j) = h_{\mathrm{MB}}(j) := \frac{1}{\theta}\Gamma\left(\frac{j+1+\alpha}{\theta}\right), \quad j_* = \max\{\lceil 2\theta-1\rceil, \lceil 2\theta-1-\alpha\rceil\}. \tag{4.4.6}$$

The respective scaling limits $P^{(\infty)}$ and $Q^{(\infty)}$ (cf. Proposition 4.3.10) are given in terms of the Wright's generalization of the Bessel function

$$J_{a,b}(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\,\Gamma(aj+b)}. \tag{4.4.7}$$

Explicitly,

$$P^{(\infty)}(y) = \theta J_{\frac{1}{\theta},\frac{\alpha+1}{\theta}}(y), \quad Q^{(\infty)}(x) = x^{\alpha} J_{\theta,\alpha+1}(x^{\theta}), \tag{4.4.8}$$

From Proposition 4.3.10 one recovers the known limiting kernel [23]

$$K_{\mathrm{MB}}^{(\infty)}(x,y) = \theta \int_0^1 dt (xt)^{\alpha} J_{\theta,\alpha+1}((xt)^{\theta}) J_{\frac{1}{\theta},\frac{\alpha+1}{\theta}}(yt). \tag{4.4.9}$$

One can then use Theorem 4.3.8, by simply plugging the limiting kernel in (4.2.12), to obtain the $1,k$-point function, or, equivalently, in (4.2.18) to get the $1,k$-cross-covariance density.

Alternatively, one can use Corollary 4.3.15 to obtain the $1,k$-cross-covariance and consequently, the $1,k$-point function, with

$$\rho_{EV}^{(\infty)}(x) = \theta x^{\alpha} e^{-x^{\theta}} E_{\frac{1}{\theta},\frac{\alpha+1}{\theta}}(x), \tag{4.4.10}$$

$$H_0^{(\infty)}(x,y) = \theta \int_0^1 du\,(yu)^{\alpha} J_{\theta,\alpha+1}\left((yu)^{\theta}\right) E_{\frac{1}{\theta},\frac{\alpha+1}{\theta}}(xu), \tag{4.4.11}$$

$$V_1^{(\infty)}(x,y) = \theta \int_0^1 du\, J_{\frac{1}{\theta},\frac{\alpha+1}{\theta}}(yu)\,(xu)^{\alpha} e^{-(xu)^{\theta}}, \tag{4.4.12}$$

where $E_{a,b}$ is the Mittag-Leffler function

$$E_{a,b}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}, \quad a > 0. \tag{4.4.13}$$

This second formulation of the $1,k$-point function (and the $1,k$-cross-covariance) has the advantage of involving a single compact integral and is numerically much more efficient to use. One can even go further and use the explicit expressions of the Wright generalized Bessel function (4.4.7) and the Mittag-Leffler function (4.4.13) to compute the integrals $H_0^{(\infty)}(x,y)$ and $V_1^{(\infty)}(x,y)$ term by term, in a similar way as in the proof of Corollary 3.4.3.

Finally, one can check that Assumptions 4.2.1 are verified as well when $\alpha \geqslant 0$. Explicitly, using the asymptotic expansion of the Wright's function (4.4.7) [151], the functions f and g can be taken to be

$$g(y) = |y| + c_0, \qquad f(x) = c_1 x^{\frac{1}{1+\theta}} + c_2, \quad c_0, c_1, c_2 > 0, \tag{4.4.14}$$

and the last assumption is checked in section 4.5.

4.5 Consistency check of assumptions

In this section, we will verify that the technical assumption (4.2.8) made to control the tail of the polynomials—in the general setting of polynomial ensembles, is not more restrictive than Assumptions 4.3.1—in the setting of the Pólya subclass. In other words, we want to check that the set of Pólya ensemble verifying Assumptions 4.3.1 is still potentially[6] a subset of the polynomial ensembles verifying Assumptions 4.2.1. To this end, it is sufficient to show that Assumptions 4.3.1 imply (4.2.8). This is indeed the case, as shown by the following Lemma.

Lemma 4.5.1 *Let $n \in \mathbb{N}$, $n > 2$, $x \in \mathbb{R}$, $R > 0$. For a Pólya ensemble verifying Assumptions 4.3.1 with polynomials $\{p_j\}_{j=0,\dots,n-1}$ given by*

$$p_j\left(\frac{x}{\nu_n}\right) = \sum_{c=0}^{j} \binom{j}{c} \frac{(-x)^c}{\xi_n \nu_n^c \mathcal{M}w_n(c+1)}, \tag{4.5.1}$$

we have

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_R^\infty dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \frac{1}{n} \sum_{j=0}^{n-1} \left| p_j\left(\frac{xt}{n\nu_n}\right)\right| = 0. \tag{4.5.2}$$

We stress here that the particular choice of the polynomials p_j (4.5.1) is done without loss of generality as the expression of the p_j is not uniquely given; cf. Remark 2.1.12. The choice (4.5.1) corresponds to the expression (2.1.38) with an additional factor $1/\xi_n$.

Proof of Lemma 4.5.1. The idea of the following computation is to rewrite the sum of polynomials as another polynomial and use the Assumptions 4.3.1 to show the non-vanishing contribution of the integrand in (4.5.2) remains in a bounded interval close to the origin as $n \to \infty$. The assumption (4.3.4) excludes a fixed number of the first summands. This is not an issue as one can split the sum in two parts and deal with this fix number of summands separately. Then, one can compute the t integral explicitly in terms of an incomplete beta functions and treat the different regimes depending of the indices j.

Let us start by denoting

$$\alpha_{c,n} := \xi_n \nu_n^c \mathcal{M}w_n(c+1). \tag{4.5.3}$$

We then use the summation identity

$$\sum_{j=0}^{n-1}\sum_{k=0}^{j} a_{k,j} = \sum_{j=0}^{n-1}\sum_{c=j}^{n-1} a_{j,c} \tag{4.5.4}$$

[6] The restrictivity of (4.2.6) being unclear.

to express the sum of polynomials as another polynomial

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|p_j\left(\frac{x}{n\nu_n}\right)\right| \leqslant \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{j}{k}\frac{|x|^k}{n^k|\alpha_{k,n}|} = \sum_{j=0}^{n-1}\left(\frac{1}{n}\sum_{c=j}^{n-1}\binom{c}{j}\right)\frac{|x|^j}{n^j|\alpha_{j,n}|}. \quad (4.5.5)$$

Using the following bound,

$$\frac{1}{n}\sum_{c=j}^{n-1}\binom{c}{j} = \binom{n-1}{j}\frac{1}{j+1} \leqslant \binom{n-1}{j}, \quad (4.5.6)$$

one gets to

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|p_j\left(\frac{xt}{n\nu_n}\right)\right| \leqslant \sum_{j=0}^{n-1}\binom{n-1}{j}\frac{|xt|^j}{n^j|\alpha_{j,n}|}. \quad (4.5.7)$$

Turning to the integral, we then get the following bound after proceeding with the change of variable $t \mapsto nt$,

$$\int_R^\infty dt\frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}}\frac{1}{n}\sum_{j=0}^{n-1}\left|p_j\left(\frac{xt}{n\nu_n}\right)\right| \leqslant S_n(R) + T_n(R), \quad (4.5.8)$$

with

$$S_n(R) := 2n^2\sum_{j=0}^{\lfloor R\rfloor-1}\int_{\frac{R}{n}}^\infty dt\frac{t}{(1+t)^{n+2}}\binom{n-1}{j}\frac{|xt|^j}{|\alpha_{j,n}|}, \quad (4.5.9)$$

and

$$T_n(R) := 2n^2\sum_{j=\lfloor R\rfloor}^{n-1}\int_{\frac{R}{n}}^\infty dt\frac{t}{(1+t)^{n+2}}\binom{n-1}{j}\frac{|xt|^j}{h(j)}. \quad (4.5.10)$$

We have introduced, here, a cut-off at $j = \lfloor R\rfloor$, which serves to distinguish between two different asymptotic regimes. Note that we have used assumption (4.3.4) to get (4.5.10). Indeed, as j_* is fixed one can always choose R big enough such that $R > j_*$.

Computation of the t integral

The t integral can be computed term by term and yields an incomplete beta function

$$\int_{\frac{R}{n}}^\infty dt\frac{t^{j+1}}{(1+t)^{n+2}} = \mathrm{B}\left(\frac{n}{n+R}; n-j, j+2\right) = \int_0^{\frac{n}{n+R}} dt\, t^{n-j-1}(1-t)^{j+1}. \quad (4.5.11)$$

To extract the R dependence which will be crucial in the coming analysis, one needs to distinguish two different regimes as j can grow with n. Hence the arbitrary choice of the cutoff $j = \lfloor R\rfloor$.

The first regime is for $j \in [\![0, \lfloor R\rfloor - 1]\!]$. As j does not grow with n here, one can

use the asymptotic expansion of the beta function, given in [104],

$$\mathrm{B}\left(\frac{n}{n+R}; n-j, j+2\right) \underset{n\to\infty}{=} \frac{1}{n-j+1}\left(1-\frac{R}{n+R}\right)^{n-j}\left(\frac{R}{n+R}\right)^{j+1} \times\left[1+O\left(\frac{(j+1)!}{n-j+1}\right)\right]. \tag{4.5.12}$$

For the second regime, $j \in [\![\lfloor R\rfloor - 1, n-1]\!]$, one can simply use the identity

$$\binom{n-1}{j} = \frac{j+1}{n(n+1)}\frac{1}{\mathrm{B}(n-j, j+2)} \tag{4.5.13}$$

to get the following bound

$$\mathrm{B}\left(\frac{n}{n+R}; n-j, j+2\right) \leqslant \mathrm{B}(n-j, j+2) = \frac{j+1}{n(n+1)}\frac{1}{\binom{n-1}{j}}. \tag{4.5.14}$$

Computation of $S_n(R)$

Let us first focus on $S_n(R)$ (4.5.9). The integral is given by the beta function (4.5.11) and one can split the sum in two parts to use assumption (4.3.4) and get the bound

$$S_n(R) \leqslant S_n^{(1)}(R) + S_n^{(2)}(R), \tag{4.5.15}$$

with

$$\begin{aligned} S_n^{(1)}(R) &:= 2n^2 \sum_{j=0}^{j_*-1} \binom{n-1}{j}\frac{|x|^j}{|\alpha_{j,n}|}\mathrm{B}\left(\frac{n}{n+R}; n-j, j+2\right), \\ S_n^{(2)}(R) &:= 2n^2 \sum_{j=j_*}^{\lfloor R\rfloor-1} \binom{n-1}{j}\frac{|x|^j}{h(j)}\mathrm{B}\left(\frac{n}{n+R}; n-j, j+2\right). \end{aligned} \tag{4.5.16}$$

As all indices are in the interval $[0, R-1]$ one can use the expansion (4.5.12). Combining it with the fact

$$\binom{n-1}{j}\frac{1}{n^j} \leqslant \frac{1}{j!}, \tag{4.5.17}$$

along with the bound assumption (4.3.2) on $S_n^{(1)}(R)$, one gets, for all n large enough,

$$S_n^{(1)}(R) \leqslant C_1 R\left(1-\frac{R}{n+R}\right)^{n-j_*+1}\sum_{j=0}^{j_*-1}\frac{(R|x|)^j}{j!}, \tag{4.5.18}$$

$$S_n^{(2)}(R) \leqslant C_2 R\left(1-\frac{R}{n+R}\right)^{n-\lfloor R\rfloor+1}\sum_{j=j_*}^{\lfloor R\rfloor-1}\frac{(R|x|)^j}{j!\,h(j)}, \tag{4.5.19}$$

where C_1, C_2 are some positive constants independent of n and R. As j_* is n-independent, taking the limit $n \to \infty$ on both sides of the two inequalities above

yields

$$\lim_{n\to\infty} S_n^{(1)}(R) + S_n^{(2)}(R) \leqslant C_1 R e^{-R} \sum_{j=0}^{j_*-1} \frac{(R|x|)^j}{j!} + C_2 R e^{-R} \sum_{j=j_*}^{\lfloor R\rfloor -1} \frac{(R|x|)^j}{j!\, h(j)}. \tag{4.5.20}$$

Let us show the RHS of the above inequality vanishes in the limit $R \to \infty$. Making use of Stirling approximation to get a rough lower bound of the factorial,

$$\left(\frac{j}{e}\right)^j \leqslant \sqrt{2\pi j}\left(\frac{j}{e}\right)^j \leqslant \Gamma(j+1), \quad j \geqslant 1, \tag{4.5.21}$$

one gets the following inequalities

$$e^{-R} \sum_{j=j_*}^{\lfloor R\rfloor -1} \frac{(R|x|)^j}{j!\, h(j)} \leqslant e^{-R/2} \sum_{j=j_*}^{\lfloor R\rfloor -1} \left(\frac{R}{j}\exp\left(-\frac{R}{2j}\right)|x|e\right)^j \frac{1}{h(j)} \leqslant e^{-R/2} \sum_{j=j_*}^{\infty} \frac{(2|x|)^j}{h(j)}, \tag{4.5.22}$$

where we have used the fact

$$y\exp(-y/2) \leqslant 2e^{-1}, \quad \forall y \geqslant 0. \tag{4.5.23}$$

Note that the RHS of (4.5.22) is indeed finite due to the root asymptotic of h (4.3.5), which guarantees infinite radius of convergence of the series, by Cauchy-Hadamard theorem. Moreover, as j_* is independent of R, the series itself is independent of R and we have

$$\lim_{R\to\infty} \left(C_1 R e^{-R} \sum_{j=0}^{j_*-1} \frac{(R|x|)^j}{j!} + e^{-R/2} \sum_{j=j_*}^{\infty} \frac{(2|x|)^j}{h(j)} \right) = 0, \tag{4.5.24}$$

which finishes to show $\lim_{R\to\infty}\lim_{n\to\infty} S_n(R) = 0$.

Computation of $T_n(R)$

Turning to $T_n(R)$ (4.5.10), in this case, all the indices are in the interval $[R, n-1]$. Therefore, one can use the bound (4.5.14) to get

$$T_n(R) \leqslant 2 \sum_{j=\lfloor R\rfloor}^{n-1} (j+1)\frac{|x|^j}{h(j)}. \tag{4.5.25}$$

Due to the root asymptotic of h (4.3.5), by Cauchy-Hadamard theorem, the RHS is a convergent series as $n \to \infty$. Hence,

$$\lim_{n\to\infty} T_n(R) \leqslant 2 \sum_{j=\lfloor R\rfloor}^{\infty} (j+1)\frac{|x|^j}{h(j)}. \tag{4.5.26}$$

The RHS of the above inequality being the remainder of a convergent series, taking the limit $R \to \infty$ finishes to show $\lim_{R\to\infty}\lim_{n\to\infty} T_n(R) = 0$.

■

4.6 Proof: Hard edge at the origin

4.6.1 Proof of Theorem 4.2.5

The proof of Theorem 4.2.5 relies on Lebesgue's dominated convergence theorem, hence Assumptions 4.2.1. However, we only assume the existence of a continuous bound, not necessarily in $\mathrm{L}^1(\mathbb{R}_+)$. One has, therefore, to split the integration into two parts: a compact part where one can use bounded convergence theorem to bring the limit inside the integral and a non-compact part which vanishes in the appropriate limits. The most important step of the proof relies therefore on the following Lemma.

Lemma 4.6.1 *Let $k \in \mathbb{N}$. Under Assumptions 4.2.1, for almost all $r \geqslant 0$, there exists a positive constant C_R, n-independent, such that $\lim_{R\to\infty} C_R = \infty$ and for which we have*

$$\lim_{R\to\infty}\lim_{n\to\infty}\left(\int_0^R dt\int_{C_R}^n dv+\int_R^\infty dt\int_0^n dv\right)\frac{n^2}{v}\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\times\det\begin{pmatrix}\frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{a_c}{n\nu_n}\right)-\delta(rv-a_c)\\ \frac{1}{n\nu_n}K_n\left(\frac{a_b}{n\nu_n},\frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n}K_n\left(\frac{a_b}{n\nu_n},\frac{a_c}{n\nu_n}\right)\end{pmatrix}_{b,c=1}^{k}=0. \tag{4.6.1}$$

Expanding the determinant in (4.6.1) one can see there are 3 different types of integrands, each time integrated on 2 different domains. We therefore introduce the following notations for the 6 different quantities[7]:

$$\begin{aligned} I^-_{0,n}(R) &:= \int_R^\infty dt\int_0^n \frac{dv}{v}n^2\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\frac{1}{n\nu_n}K_n\left(\frac{a_1}{n\nu_n},\frac{-rt}{n\nu_n}\right)\delta(rv-a_2),\\ I^+_{0,n}(R) &:= \int_0^R dt\int_{C_R}^n \frac{dv}{v}n^2\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\frac{1}{n\nu_n}K_n\left(\frac{a_1}{n\nu_n},\frac{-rt}{n\nu_n}\right)\delta(rv-a_2), \end{aligned} \tag{4.6.2}$$

$$\begin{aligned} I^-_{1,n}(R) &:= \int_R^\infty dt\int_0^n \frac{dv}{v}n^2\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{-rt}{n\nu_n}\right),\\ I^+_{1,n}(R) &:= \int_0^R dt\int_{C_R}^n \frac{dv}{v}n^2\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{-rt}{n\nu_n}\right), \end{aligned} \tag{4.6.3}$$

[7] We highlight the main differences in color.

and

$$I_{2,n}^{-}(R) := \int_R^\infty dt \int_0^n \frac{dv}{v} n^2 \varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) \frac{1}{(n\nu_n)^2} K_n\left(\frac{rv}{n\nu_n}, \frac{a_1}{n\nu_n}\right) K_n\left(\frac{a_2}{n\nu_n}, \frac{-rt}{n\nu_n}\right),$$
$$I_{2,n}^{+}(R) := \int_0^R dt \int_{C_R}^n \frac{dv}{v} n^2 \varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) \frac{1}{(n\nu_n)^2} K_n\left(\frac{rv}{n\nu_n}, \frac{a_1}{n\nu_n}\right) K_n\left(\frac{a_2}{n\nu_n}, \frac{-rt}{n\nu_n}\right). \tag{4.6.4}$$

The proof of Lemma 4.6.1 relies on the more elementary lemma which follows.

Lemma 4.6.2 *Let $n \in \mathbb{N}$, $n > 2$, and a continuous function $f : \mathbb{R}_+^* \to \mathbb{R}_+$ such that $f(u) = o(u)$, and $u \mapsto \exp(f(u)) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$. There exists $F : \mathbb{R}_+ \to \mathbb{R}_+$ a n-independent, continuous and strictly decreasing function such that $F(x) \to 0$, as $x \to +\infty$, and that we have*

$$\int_x^n dv(1+v)\left(1-\frac{v}{n}\right)^{n-2} \exp(f(v)) \leqslant F(x), \quad \forall x \in [0,n]. \tag{4.6.5}$$

Proof. Using the fact

$$\forall v \in [0,n], \quad \left(1-\frac{v}{n}\right)^{n-2} \leqslant 2e^{-v}, \tag{4.6.6}$$

one has

$$\begin{aligned}\int_x^n dv\,(1+v)\left(1-\frac{v}{n}\right)^{n-2} \exp(f(v)) &\leqslant 2\int_x^n dv\,(1+v)\exp(-v+f(v)) \\ &\leqslant 2\int_x^\infty dv\,(1+v)\exp(-v+f(v)).\end{aligned} \tag{4.6.7}$$

Due to the asymptotic behaviour of f, $f(u) = o(u)$, as $u \to \infty$,

$$F(x) := 2\int_x^\infty dv\,(1+v)\exp(-v+f(v)) \tag{4.6.8}$$

is the remainder of a converging integral of a continuous integrand on $\mathbb{R}_+^*$. Moreover, $F(0)$ is well defined due to $u \mapsto \exp(f(u)) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$. Therefore F is continuous on $\mathbb{R}_+$ and $F(x) \to 0$, as $x \to +\infty$. Moreover, as the integrand is strictly positive, we have $F' < 0$, which implies F is strictly decreasing. ∎

Proof of Lemma 4.6.1. Let us start with $I_{1,n}^{-}(R)$. The first step is to bound φ_n. Explicitly, we have

$$\frac{n^2}{v}\varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) = \left(t+v-1-\frac{1}{n^2}vt\right)\left(1-\frac{v}{n}\right)^{n-2}\left(1+\frac{t}{n}\right)^{-(n+2)}. \tag{4.6.9}$$

Thus, we can choose the following bound

$$\forall (v,t) \in [0,n] \times \mathbb{R}_+, \quad \left| \frac{n^2}{v} \varphi_n \left(\frac{v}{n}, \frac{t}{n} \right) \right| \leqslant (1+v) \left(1 - \frac{v}{n}\right)^{n-2} \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}}. \tag{4.6.10}$$

Regarding the kernel, using the expression (2.1.24) with the assumption (4.2.6) yields, for all v and t,

$$\left| \frac{1}{n\nu_n} K_n \left(\frac{rv}{n\nu_n}, \frac{-rt}{n\nu_n} \right) \right| \leqslant \frac{\exp(f(rv))}{n} \sum_{j=0}^{n-1} \left| p_j \left(-\frac{rt}{n\nu_n} \right) \right|, \tag{4.6.11}$$

where $f(u) = o(u)$, as $u \to \infty$, and $u \mapsto \exp(f(u)) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$. From there, we get

$$\begin{aligned} &\left| I^-_{1,n}(R) \right| \\ &\leqslant \int_R^\infty dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \frac{1}{n} \sum_{j=0}^{n-1} \left| p_j \left(-\frac{rt}{n\nu_n} \right) \right| \int_0^n dv (1+v) \left(1 - \frac{v}{n}\right)^{n-2} \exp(f(rv)). \end{aligned} \tag{4.6.12}$$

As r is fixed, using Lemma 4.6.2 yields

$$\int_0^n dv (1+v) \left(1 - \frac{v}{n}\right)^{n-2} \exp(f(rv)) \leqslant F(0) = C, \tag{4.6.13}$$

for some positive constant C, independent of n. Finally, using assumption (4.2.8) finishes to show that $\lim_{R\to\infty} \lim_{n\to\infty} I^-_{1,n}(R) = 0$.

Turning to $I^+_{1,n}(R)$, in a similar way, one has

$$\begin{aligned} &\left| I^+_{1,n}(R) \right| \\ &\leqslant \int_0^R dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \frac{1}{n} \sum_{j=0}^{n-1} \left| p_j \left(-\frac{rt}{n\nu_n} \right) \right| \int_{C_R}^n dv (1+v) \left(1 - \frac{v}{n}\right)^{n-2} \exp(f(rv)). \end{aligned} \tag{4.6.14}$$

Using Lemma 4.6.2 and (4.2.7) leads to

$$\left| I^+_{1,n}(R) \right| \leqslant F(C_R) \int_0^R dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \exp\left(g(-rt)\right). \tag{4.6.15}$$

Recalling that r is fixed, one can always find a constant S_R independent of n to get the very loose bound

$$\int_0^R dt \frac{(1+t)}{\left(1+\frac{t}{n}\right)^{n+2}} \exp\left(g(-rt)\right) \leqslant R \sup_{t\in[0,R]} \exp\left(g(-rt)\right) \leqslant S_R, \tag{4.6.16}$$

where we have used that $(1+t)/\left(1+\frac{t}{n}\right)^{n+2} \leqslant 1$, for all $t \geqslant 0$. Then, since F is strictly decreasing, it is invertible. Moreover, because $F(x) \to 0$ as $x \to \infty$, for all

sufficiently large R such that $\exp(-R-S_R)\in F(\mathbb{R}_+)$, one can set

$$C_R = F^{-1}\left(\exp(-R-S_R)\right). \tag{4.6.17}$$

As, $F(x)\to 0$ as $x\to\infty$, it follows that $C_R\to\infty$ as $R\to\infty$. This finishes to show that $\lim_{R\to\infty}\lim_{n\to\infty} I^{+}_{1,n}(R)=0$. Note, here, the need to choose the domains of the remainder integrals carefully (4.6.22).

Moving to $I^{-}_{2,n}(R)$ and $I^{+}_{2,n}(R)$, as a_1, a_2 are fixed, using once again the expression (2.1.24) with (4.2.6) yields, for all v in compact subsets of $\mathbb{R}_+$ and for all $t\in\mathbb{R}$,

$$\begin{aligned}&\left|\frac{1}{(n\nu_n)^2}K_n\left(\frac{rv}{n\nu_n},\frac{a_1}{n\nu_n}\right)K_n\left(\frac{a_2}{n\nu_n},\frac{-rt}{n\nu_n}\right)\right|\\ &\leqslant \exp(g(a_1)+f(a_2)+f(rv))\frac{1}{n}\sum_{j=0}^{n-1}\left|p_j\left(-\frac{rt}{n\nu_n}\right)\right|.\end{aligned} \tag{4.6.18}$$

This is almost the same upper bound as (4.6.11) where the only difference is the multiplicative constant $\exp(g(a_1)+f(a_2))$ which is independent of n and R. Thus, we also have $\lim_{R\to\infty}\lim_{n\to\infty} I^{\pm}_{2,n}(R)=0$.

Let us now consider $I^{\pm}_{0,n}(R)$. We immediately have $\lim_{R\to\infty}\lim_{n\to\infty} I^{+}_{0,n}(R)=0$. Indeed, applying the Dirac delta functions gives

$$I^{+}_{0,n}(R) := \Theta(n-a_2)\Theta(a_2-C_R)\int_0^R dt\frac{n^2}{a_2}\varphi_n\left(\frac{a_2}{rn},\frac{t}{n}\right)\frac{1}{n\nu_n}K_n\left(\frac{a_1}{n\nu_n},\frac{-rt}{n\nu_n}\right) \tag{4.6.19}$$

and, as a_2 is fixed, $I^{+}_{0,n}(R)$ vanishes for $C_R>a_2$, due to the Heaviside step function. Regarding $I^{-}_{0,n}(R)$, one follows the same steps as for the case $I^{-}_{1,n}(R)$, using assumption (4.2.8), this time without the v integral, and gets $\lim_{R\to\infty}\lim_{n\to\infty} I^{-}_{0,n}(R)=0$.

Finally, to finish the proof, one expands the determinant in (4.6.1) using Leibniz formula, uses the finite limit assumption (4.2.5) for the terms where none of the arguments are integrated and uses the vanishing limits of $I^{\pm}_{0,n}(R)$, $I^{\pm}_{1,n}(R)$ and $I^{\pm}_{2,n}(R)$ computed above. ■

Proof of Theorem 4.2.5. Starting from (3.3.10), proceeding with the scaling of the variables and doing the following change of variables $t\mapsto\frac{t}{n}$ and $v\mapsto\frac{rv}{n\nu_n}$ yields

$$\begin{aligned}&\frac{n^{k+1}}{\nu_n(n\nu_n)^k}f_{1,k}\left(\frac{r}{\nu_n};\frac{a_1}{n\nu_n},\dots,\frac{a_k}{n\nu_n}\right)\\ &\quad=\frac{n^k(n-k)!}{n!}\int_0^\infty dt\int_0^n\frac{dv}{v}n^2\varphi_n\left(\frac{v}{n},\frac{t}{n}\right)\\ &\quad\times\det\begin{pmatrix}\frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n}K_n\left(\frac{rv}{n\nu_n},\frac{a_c}{n\nu_n}\right)-\delta(rv-a_c)\\ \frac{1}{n\nu_n}K_n\left(\frac{a_b}{n\nu_n},\frac{-rt}{n\nu_n}\right) & \frac{1}{n\nu_n}K_n\left(\frac{a_b}{n\nu_n},\frac{a_c}{n\nu_n}\right)\end{pmatrix}_{b,c=1}^{k},\end{aligned} \tag{4.6.20}$$

where we have used the scaling property of the Dirac delta function

$$\forall c \neq 0, \quad \delta\left(\frac{x}{c}\right) = c\,\delta(x). \tag{4.6.21}$$

Let us now split the domain of integration $\mathcal{D}_t$ of the t integral (resp. $\mathcal{D}_v$ for the v integral) in two parts: one on which all the limits exist and one on which the integral will vanish as $n \to \infty$ then $R \to \infty$. The appropriate choice given Assumptions 4.2.1 is the following

$$\mathcal{D}_t = [0, R] \cup [R, \infty[, \quad \mathcal{D}_v = [0, C_R] \cup [C_R, n], \tag{4.6.22}$$

where C_R is the constant in Lemma 4.6.1 corresponding to the bound f in (4.2.6). The choice of C_R is only a technical details needed in the proof of Lemma 4.6.1 (cf. (4.6.15)). Coming back to the integrand, we explicitly have

$$n^2\varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) = v\left(t + v - 1 - \frac{1}{n^2}vt\right)\left(1 - \frac{v}{n}\right)^{n-2}\left(1 + \frac{t}{n}\right)^{-(n+2)} \tag{4.6.23}$$

and for any $(v, t) \in [0, C_R] \times [0, R]$, where R is fixed, we have the uniform convergence of the limit

$$\lim_{n\to\infty} n^2\varphi_n\left(\frac{v}{n}, \frac{t}{n}\right) = v(t + v - 1)e^{-v}e^{-t} = \varphi^{(\infty)}(v, t). \tag{4.6.24}$$

Using the pointwise convergence of the sequence of scaled kernels (4.2.5) and the bounds of the p_j (4.2.7) and q_j (4.2.6) one can use bounded convergence theorem to take the limit $n \to \infty$, below the integrals. Using the limit,

$$\lim_{n\to\infty} \frac{n^k(n-k)!}{n!} = 1, \tag{4.6.25}$$

for fixed k and then taking the limit $R \to \infty$, yields (4.2.12). Using Lemma 4.6.1 finishes to prove the remainder vanishes in the large n, then R, limits. ■

4.6.2 Proof of Theorem 4.3.8

The idea of the proof of Theorem 4.3.8 is to use Corollary 4.3.15, which is proven independently of Theorem 4.3.8, and reintroduce the integrals in (4.3.36) to get back to the integral formulation (4.2.12).

In addition to Assumptions 4.3.1, one needs to assume that $K^{(\infty)}$ grows at most sub-exponentially in the first argument, which is sufficient to guarantee absolute convergence of all integrals encountered. We strongly believe absolute convergence of (4.2.12) should hold under Assumptions 4.3.1, however it is not shown as the bound (4.3.20) only holds for bounded intervals.

To motivate this, one can notice (4.2.12) is indeed absolutely convergent for all classical ensembles presented in this thesis, as the limiting kernel is either the Bessel kernel (4.3.32) or the Muttalib-Borodin kernel (4.4.9), which are both integrable in

the first argument, without the help of the decreasing exponential.

Therefore, unless the limiting kernel exhibits oscillations in the first argument, whose amplitudes are growing exponentially, the absolute convergence of (4.2.12) should be guaranteed by the decreasing exponential contained in $\varphi^{(\infty)}$ (4.2.13).

We begin by proving the following lemma, introduced for convenience, to handle a recurring type of integral that arises in the proof of Theorem 4.3.8.

Lemma 4.6.3 *Given $\varphi^{(\infty)}$ defined in (4.2.13) and $K^{(\infty)}$ (4.2.5), under Assumptions 4.3.1 and further assuming that $x \mapsto K^{(\infty)}(x, y)$ grows at most sub-exponentially as $x \to \infty$, uniformly in $y \in \mathbb{R}$, we have*

$$\int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}(v,t)\, K^{(\infty)}(rv,-rt) = \partial_r \int_0^\infty dt \int_0^\infty dv\, e^{-v} e^{-t} r K^{(\infty)}(rv,-rt) \tag{4.6.26}$$

and

$$\begin{aligned} &\int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}\left(\tfrac{v}{r}, t\right) K^{(\infty)}(a_b,-rt)\left[\delta(v-a_c) - K^{(\infty)}(v,a_c)\right] \\ &= \partial_r \int_0^\infty dt \int_0^\infty \frac{dv}{r} e^{-\frac{v+t}{r}} K^{(\infty)}(a_b,-t)\left[\delta(v-a_c) - K^{(\infty)}(v,a_c)\right]. \end{aligned} \tag{4.6.27}$$

Proof. Let us first note that all integrals in Lemma 4.6.3 are absolutely convergent for any $r \in \mathbb{R}_+$. This is guaranteed by the exponential decay of $\varphi^{(\infty)}$ (4.2.13). Indeed, we assume that $K^{(\infty)}$ grows at most sub-exponentially in the first argument and the last of Assumptions 4.3.1 implies the asymptotic

$$P^{(\infty)}(|x|) = \exp(o(|x|)), \quad |x| \to \infty, \tag{4.6.28}$$

as it can be seen in (4.5.22). The explicit computation of those integrals is the object of the proof of Theorem 4.3.8.

Starting with (4.6.26), proceeding with a change of variables $t \mapsto t/r$ and $v \mapsto v/r$, one can apply Leibniz integral rule, due to the absolute convergence, and this yields

$$\begin{aligned} &\partial_r \int_0^\infty dt \int_0^\infty dv\, e^{-v} e^{-t} r K^{(\infty)}(rv,-rt) \\ &= \int_0^\infty dt \int_0^\infty dv\, e^{-\frac{v+t}{r}} \left(\frac{t+v}{r^3} - \frac{1}{r^2}\right) K^{(\infty)}(v,-t). \end{aligned} \tag{4.6.29}$$

Doing the inverse change of variables $t \mapsto rt$ and $v \mapsto rv$, on the RHS finishes to show (4.6.26). The second equality (4.6.27) is done similarly. ∎

Proof of Theorem 4.3.8. Let us consider Assumptions 4.3.1 with the extra assumption that $K^{(\infty)}$ grows at most sub-exponentially in the first argument. As Corollary 4.3.15 is proven independently of Theorem 4.3.8, we need to show that the RHS of (4.2.19) and (4.3.36) are equal and that the RHS of (3.3.13) and (4.3.39) are also

equal. Let us first show that

$$\rho_{EV}^{(\infty)}(r) = w^{(\infty)}(r) \sum_{k=0}^{\infty} \frac{r^k}{\tilde{w}^{(\infty)}(k+1)} = \int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}(v,t)\, K^{(\infty)}(rv,-rt)\,. \tag{4.6.30}$$

Using Lemma 4.6.3, along with Proposition 4.3.10, one can apply the derivative in r using a change of variable $u \mapsto u/r$ to get

$$\begin{aligned} &\int_0^\infty dt \int_0^\infty \frac{dv}{v} \varphi^{(\infty)}(v,t)\, K^{(\infty)}(rv,-rt) \\ &= \partial_r \int_0^\infty dt \int_0^\infty dv\, e^{-v} e^{-t} r \int_0^1 du\, Q^{(\infty)}(rvu) P^{(\infty)}(-rtu) \\ &= \int_0^\infty dt \int_0^\infty dv\, e^{-v} e^{-t} Q^{(\infty)}(rv) P^{(\infty)}(-rt). \end{aligned} \tag{4.6.31}$$

As the two integrals factorise, one can use the expression of $P^{(\infty)}$ (4.3.26) to compute the t integral, which is equal to $j!$. This simplifies with the factorial factor in the denominator and we get

$$\int_0^\infty dt \int_0^\infty dv\, e^{-v} e^{-t} Q^{(\infty)}(rv) P^{(\infty)}(-rt) = \int_0^\infty dv\, e^{-v} Q^{(\infty)}(rv) \sum_{j=0}^{\infty} \frac{r^j}{\tilde{w}^{(\infty)}(j+1)}. \tag{4.6.32}$$

Then, using the expression of $Q^{(\infty)}$ (4.3.27), one would want to exchange the order of integration. However, whether this is allowed is not immediately apparent. Instead we define,

$$W(r) := \int_0^\infty dv\, e^{-v} Q^{(\infty)}(rv). \tag{4.6.33}$$

The function W is well defined, as the integral is absolutely convergent, and can be seen as some multiplicative Mellin convolution between $G(x) := \exp(-x)$ and $Q^{(\infty)}$. Considering the strip $\mathbb{S}_{]1,2[} := \{z \in \mathbb{C} \mid 1 < \mathrm{Re}(z) < 2\}$, the Mellin transform on W is given by

$$\mathcal{M}W(s) = \mathcal{M}G(1-s)\, \mathcal{M}Q^{(\infty)}(s), \quad s \in \mathbb{S}_{]1,2[}, \tag{4.6.34}$$

with $\mathcal{M}G(1-s) = \Gamma(1-s)$ and

$$\mathcal{M}Q^{(\infty)}(s) = \frac{\mathcal{M}w^{(\infty)}(s)}{\Gamma(1-s)}. \tag{4.6.35}$$

Hence,

$$\mathcal{M}W(s) = \mathcal{M}w^{(\infty)}(s), \tag{4.6.36}$$

which yields, by the Mellin inversion theorem,

$$W(r) = w^{(\infty)}(r). \tag{4.6.37}$$

This finishes to show (4.6.30).

It remains to be shown that

$$\begin{aligned}&\partial_r \int_0^\infty dt \int_0^\infty \frac{dv}{r} e^{-\frac{v+t}{r}} K^{(\infty)}(a_b, -t) \left[\delta(v - a_c) - K^{(\infty)}(v, a_c)\right] \\ &= \partial_r \left[H_0^{(\infty)}(r, a_1) \left(e^{-\frac{a_2}{r}} - r V_1^{(\infty)}(r, a_2)\right)\right],\end{aligned} \tag{4.6.38}$$

with $H_0^{(\infty)}$ and $V_1^{(\infty)}$ (4.3.37) and (4.3.38). Following similar steps as the first part of the proof, one arrives at

$$\begin{aligned}&\partial_r \int_0^\infty dt \int_0^\infty \frac{dv}{r} e^{-\frac{v+t}{r}} K^{(\infty)}(a_b, -t) \left[\delta(v - a_c) - K^{(\infty)}(v, a_c)\right] \\ &= \partial_r \left[H_0^{(\infty)}(r, a_1) e^{-\frac{a_2}{r}}\right] - \partial_r \left[r H_0^{(\infty)}(r, a_1) V_1^{(\infty)}(r, a_2)\right]\end{aligned} \tag{4.6.39}$$

which is (4.3.36). Finally, (4.2.14) follows directly from Proposition 4.3.10 and this finishes the proof. ■

4.7 Summary and discussion

In the first part of this chapter, we have exploited the new results of [10], for polynomial ensembles, to study their extension to the large n limit, which are thus also new. First, we focused on the double scaling limit around the origin for the $1, k$-point correlation function between one eigenradius and k singular values as well as the $1, k$-cross-covariance density function, see Theorem 4.2.5, Corollary 4.2.7 and Corollary 4.3.15. This was motivated by the fact both singular values and eigenradii are non-negative, which implies that unless there is a growing—with n—repulsion from the origin, both ensembles should have a hard edge there. An interesting finding is the existence of a universal (at least for polynomial ensembles) scaling ratio of $1/n$ between the scale of the smallest squared singular value and scale of the smallest squared eigenradius.

Under Assumptions 4.2.1 guaranteeing the existence of the limits, the formula for the $1, k$-point correlation function involves the double scaling limit of the kernel of the corresponding determinantal point process on the singular values. The result makes clear that all ensembles for which the limiting kernel are the same will share the same limiting $1, k$-point correlation function around the origin and same limiting $1, k$-cross-covariance function (cf. (4.2.12)).

For Pólya ensembles, this translates into having same limiting Pólya weight function. As for the $1, k$-point function, it admits a more explicit form given by Corollary 4.3.15. We give the example of three classical Pólya ensembles; Laguerre, Jacobi and Cauchy-Lorentz ensembles, sharing the same limiting weight (cf. Example 4.3.4). To further illustrate this, Figure 4.1 shows the similarity between the plots of the, appropriately rescaled, cross-covariance for Jacobi and Laguerre ensembles at $n = 25$ and the plot of their shared limiting cross-covariance. The plots of Figure 4.1 also show how the deterministic constraints from Weyl's inequalities [144] translate on a probabilistic level and survive the large n limit. Here, the

plot of the cross-covariance function around the origin is reminiscent of the fact that the smallest eigenradius is bounded from below by the smallest singular value.

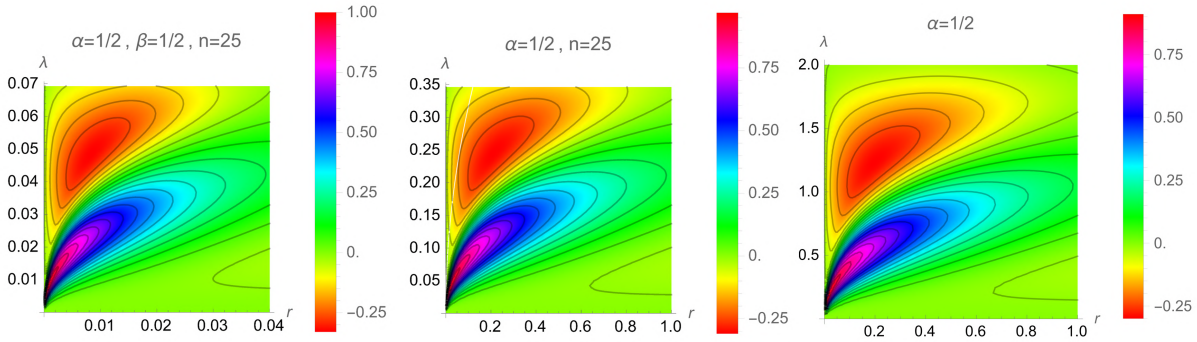


Figure 4.1: *(Left)* Plot of $(r, \lambda) \mapsto 2\lambda \mathrm{cov}(r; \lambda^2)$, for the Jacobi ensemble with parameter $n = 25$, $\alpha = 1/2$, $\beta = 1/2$.
(Middle) Plot of $(r, \lambda) \mapsto n^{3/2} 2\lambda \mathrm{cov}(r; \lambda^2)$ for the Laguerre ensemble with parameter $n = 25$, $\alpha = 1/2$.
(Right) Plot of $(r, \lambda) \mapsto 2\lambda \mathrm{cov}^{(\infty)}(r; \lambda^2)$, for $w^{(\infty)}(x) = x^{\alpha} e^{-x}$, $\alpha = 1/2$.

We then give the example of Muttalib-Borodin ensembles to emphasize how the main results Corollary 4.3.15 and Theorem 4.3.8 can be applied.

Coming back to the main results, we strongly believe the formula (4.2.12) for the limiting $1, k$-point function (equivalently, the $1, k$-cross-covariance) at the origin remains true even in the case the limiting kernel is not a function anymore but a general distribution e.g. a weighted Dirac delta function, as the formula (4.2.12) still makes sense in the distributional sense, in this case. However, proving it will require a different proof, as one would not have Lebesgue's dominated convergence theorem anymore, to interchange the limit $n \to \infty$ and the integrals. We also believe that Assumptions 4.3.1 strictly imply Assumptions 4.2.1. In section 4.5, we show that the technical restriction (4.2.8) on polynomial ensembles is always verified for Pólya ensembles verifying Assumptions 4.3.1, as well as their compositions; see Remark 4.3.5. Some assumptions can possibly be lifted with some further analysis and making use of some property of the underlying Pólya frequency functions such as their log-concavity [60, 65, 130] as well as the log-convexity of the Mellin transform of probability measures (on each subinterval of $\mathbb{R}_+$ where it is finite).

While Theorem 4.2.5 targets the largest class of ensembles where the formula (4.2.12) holds, it relies on Assumptions 4.2.1 which are hard to check in practice as they require finding the bi-orthonormal system of functions composing the kernel of the corresponding determinantal point process and this for all $n \in \mathbb{N}$. On the other hand, Theorem 4.3.8 and Corollary 4.3.15 are much more useful, as Assumptions 4.3.1 are much easier to check. Indeed, the assumptions bear only on the Pólya weight function, which is easier to find in practice.

This barely diminishes our results as the class of Pólya ensembles, albeit being smaller than the class of polynomial ensembles, is very large and comprises most of

the classical ensembles in the field of random matrices along with their compositions due to their closure property, as discussed in Chapter 1 and Remark 4.3.5.

Chapter 5

Soft-Hard edge asymptotics of correlation functions between singular values and eigenvalues for Jacobi ensembles

God forbid that Truth should be confined to mathematical demonstration.

William Blake (1757–1827)

This chapter is based on the work [8].

5.1 Preamble

In the previous chapter, we examined the collective behavior of eigenvalues and singular values in the large-n limit and on the local scale around the origin. This analysis applied to all—under reasonable assumptions—bi-unitarily invariant ensembles whose singular values follow a polynomial ensemble. In this chapter we focus on the specific case of the Jacobi ensemble (1.1.13) and look at the local scale away from the origin, around the upper hard edge for the singular values and the upper soft edge for the eigenradii. We remind that, for the Jacobi ensemble (1.1.13), the corresponding squared singular values follow the same distribution as the squared singular values of truncated unitary matrices and eigenvalues of a combination of Wishart matrices (see [51, Chapter 3]).

The main difficulty encountered in this chapter is that, contrary to the hard-edge case at the origin—where every factor in the double-integral formulation of the $1, k$-point function (4.2.3) admits a double scaling limit—at this other edge none of the terms admits an individual double scaling limit. We then proceeded with a careful and refined analysis combining known asymptotics of Jacobi polynomials in various

regimes, the use of Laplace's method, and direct integration.

5.2 Soft-hard edge scaling limit for Jacobi ensembles

While the density of most polynomial ensembles has a scaling limit at the origin, they often have only one hard edge on the singular values and eigenvalues. The Jacobi ensemble (1.1.13) with fixed parameter α and β, and corresponding Pólya weight given in (2.1.33), has a second hard edge for the singular values around the upper edge of its support. Let us study the corresponding double scaling limit around this other edge.

Using the Christoffel-Darboux formula, the kernel can be written

$$K_n(x,y) = a_n x^\alpha (1-x)^\beta \frac{P_n^{(\alpha,\beta)}(1-2x)P_{n-1}^{(\alpha,\beta)}(1-2y) - P_{n-1}^{(\alpha,\beta)}(1-2x)P_n^{(\alpha,\beta)}(1-2y)}{x-y} \tag{5.2.1}$$

with the usual Jacobi polynomials

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{z-1}{2}\right)^m \tag{5.2.2}$$

and the coefficient

$$a_n = -\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{(\alpha+\beta+2n)\Gamma(n+\alpha)\Gamma(n+\beta)}. \tag{5.2.3}$$

Remark 5.2.1 *Note that the Jacobi polynomials associated with our Jacobi ensemble are orthogonal on* $[0,1]$ *and are therefore the polynomials in* (5.2.1)*. We express them in terms of the usual Jacobi polynomials to use known asymptotics.*

One can easily show, using the reflection property of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z) \tag{5.2.4}$$

and Proposition 4.3.10, that the limiting kernel for the point process on the singular values is

$$\lim_{n\to\infty} \frac{1}{n^2} K_n\left(1-\frac{x}{n^2}, 1-\frac{y}{n^2}\right) = 4\left(\frac{x}{y}\right)^{\frac{\beta}{2}} K_\beta^{\mathrm{Bes}}(4x,4y). \tag{5.2.5}$$

Therefore, the limiting microscopic level density on the singular values is

$$\lim_{n\to\infty} \rho_{\mathrm{SV}}\left(1-\frac{a}{n^2}\right) = 4K_\beta^{\mathrm{Bes}}(4a,4a) = J_\beta\left(2\sqrt{a}\right)^2 - J_{\beta-1}\left(2\sqrt{a}\right) J_{\beta+1}\left(2\sqrt{a}\right). \tag{5.2.6}$$

However, the limit of the level density of the eigenradii around $r=1$ is trivially null

as the limiting macroscopic level density on the squared eigenradii is given by [4, Eq.(3.12)]

$$\lim_{n\to\infty} \rho_{\mathrm{EV}}(r) = \frac{1}{(1-r)^2}\Theta\left(\frac{1}{2}-r\right). \tag{5.2.7}$$

Note that the Heaviside function in (5.2.7) appears only in the limit and is not associated with a hard edge. Actually, if one zooms on a scale $1/\sqrt{n}$ around $r = 1/2$, the limiting microscopic level density of the eigenradii is given in terms of a complementary error function [4, Eq.(3.36)], defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{5.2.8}$$

While the result of [4] is for Jacobi ensembles with $\alpha = 0$ and $\beta \in \mathbb{N}$, it still holds for any real parameters α, β with $\alpha > -1$ and $\beta > 0$. Indeed, starting from the expression for the density on the eigenradii given in (2.1.47), and applying the Euler–Maclaurin formula along with Laplace's method—or equivalently, beginning with the integral representation in (3.3.13) and proceeding via a similar derivation to that in Theorem 5.2.2—one can show that

$$\frac{1}{n}\sum_{j=0}^{n-1}\frac{r^j}{\mathrm{B}(j+1+\alpha,\beta+n)} \underset{n\to\infty}{\sim} \begin{cases} \frac{1}{2}r^{-\alpha}(1-r)^{-\beta-n-1}\operatorname{erfc}\left(\frac{\sqrt{n}(2r-1)}{\sqrt{2r}}\right), & r \leqslant \frac{1}{2} \\ 2^{\alpha+\beta+2n}r^n e^{n\ln(2r)^2}\operatorname{erfc}\left(\sqrt{n}\ln(2r)\right), & r > \frac{1}{2} \end{cases} \tag{5.2.9}$$

Thus, multiplying by the Jacobi weight (2.1.33),

$$\begin{aligned}\rho_{\mathrm{EV}}(r) &= \frac{\Theta(1-r)}{n}\sum_{j=0}^{n-1}\frac{r^\alpha(1-r)^{\beta+n-1}r^j}{\mathrm{B}(j+1+\alpha,\beta+n)} \\ &\underset{n\to\infty}{\sim} \begin{cases} \frac{1}{2}(1-r)^{-2}\operatorname{erfc}\left(\frac{\sqrt{n}(2r-1)}{\sqrt{2r}}\right), & r \leqslant \frac{1}{2} \\ 2^{\alpha+\beta+2n}r^{\alpha+n}(1-r)^{\beta+n-1}e^{n\ln(2r)^2}\operatorname{erfc}\left(\sqrt{n}\ln(2r)\right), & r > \frac{1}{2}. \end{cases}\end{aligned} \tag{5.2.10}$$

Then, taking the limit $n \to \infty$, one recovers the macroscopic limiting level density (5.2.7). Alternatively, first proceeding with the scaling $r \mapsto 1/2 + \frac{r}{\sqrt{n}}$ leads to the pointwise limit

$$\lim_{n\to\infty}\rho_{\mathrm{EV}}\left(\frac{1}{2}+\frac{r}{\sqrt{n}}\right) = 2\operatorname{erfc}(2r). \tag{5.2.11}$$

There is, therefore, no other hard edge for the eigenradii than the one at the origin. The upper edge around $1/2$ is a soft edge. This is due to the n dependence of the Pólya weight, namely, the repulsion from the upper edge grows with n, in this case, for the eigenvalues, cf. (2.1.33). This also explains why both the α and β dependences drop out at the soft edge: the point $r = 1/2$ being macroscopically away from the point $r = 0$, its repulsion—whose strength depends on α—is not felt and, as the repulsion from the point $r = 1$ grows with n, the dependence of β becomes completely negligible. This confirms, heuristically, why the result of [4] is still valid for any $\alpha > -1$ and $\beta > 0$.

The $1, k$-cross-covariance density then admits a scaling limit around the soft edge

of the eigenradii and the upper hard edge of the singular values given in the following theorem. We will refer to this double scaling limit as Soft-Hard (SH) edge limit.

Theorem 5.2.2 *For the Jacobi ensemble with Pólya weight* $w_{\mathrm{Jac}}(x) = x^{\alpha}(1-x)^{\beta+n-1}\Theta(1-x)$, *with* α, β *fixed and* $\alpha > -1$, $\beta > 0$, *the pointwise limit*

$$\mathrm{cov}^{(\infty)}_{1,k\,\mathrm{SH}}(r; a_1, \dots, a_k) := \lim_{n\to\infty} n^{3/2-k}\,\mathrm{cov}_{1,k}\left(\frac{1}{2} - \frac{r}{\sqrt{n}}; 1 - \frac{a_1}{n^2}, \dots, 1 - \frac{a_k}{n^2}\right) \tag{5.2.12}$$

of the $1,k$*-cross-covariance density functions between one squared eigenradius and* k *squared singular values is given, for all fixed* $(r, a_1, \dots, a_k) \in \mathbb{R} \times \mathbb{R}_+^k$, *by*

$$\begin{aligned}
&\mathrm{cov}^{(\infty)}_{1,k\,\mathrm{SH}}(r; a_1, \dots, a_k) \\
&\quad = \frac{4}{\sqrt{\pi}} e^{-4r^2} \partial_\mu \det\left[\int_0^1 dt J_\beta(2\sqrt{a_b t}) J_\beta(2\sqrt{a_c t}) + \mu J_\beta(2\sqrt{a_b}) J_\beta(2\sqrt{a_c})\right]_{b,c=1}^{k}\Bigg|_{\mu=0} \\
&\quad = -\frac{4^k}{\sqrt{\pi}} e^{-4r^2} \det\begin{pmatrix} 0 & J_\beta(2\sqrt{a_c}) \\ J_\beta(2\sqrt{a_b}) & K_\beta^{\mathrm{Bes}}(4a_b, 4a_c) \end{pmatrix}_{b,c=1}^{k}.
\end{aligned} \tag{5.2.13}$$

In particular, in the case $k = 1$, *the limiting cross-covariance reads*

$$\mathrm{cov}^{(\infty)}_{\mathrm{SH}}(r; a) = \frac{4}{\sqrt{\pi}} e^{-4r^2} J_\beta(2\sqrt{a})^2. \tag{5.2.14}$$

Remark 5.2.3 *Echoing Remark 4.2.6, the above result implies*

$$\frac{1}{n^{2k+1/2}}\,\mathrm{cov}_{1,k}\left(\frac{1}{2} - \frac{r}{\sqrt{n}}; 1 - \frac{a_1}{n^2}, \dots, 1 - \frac{a_k}{n^2}\right) \underset{n\to\infty}{=} \frac{1}{n^{k+2}}\left[\mathrm{cov}^{(\infty)}_{1,k\,\mathrm{SH}}(r; a_1, \dots, a_k) + o(1)\right], \tag{5.2.15}$$

which means that the scaling limit of the $1,k$*-cross-covariance, for Jacobi ensembles, at the soft-hard edge is of order* $O(1/n^{k+2})$ *as* $n \to \infty$. *It is therefore a factor* $1/n$ *smaller than the limiting* $1,k$*-point function and can be seen as the first correction term.*

It is interesting to see that the formula of the limiting $1,k$-cross-covariance density at the soft-hard edge (5.2.13) involves some Bessel function for the singular value part, as one would expect from the usual universal behaviour at the hard edge for singular values (cf. (2.2.21)), and is Gaussian in the eigenradius. The latter is not so easy to foresee. However, this is to put in relation to a recurring phenomenon for β-ensembles, as pointed out in [55], that the leading correction term (cf. Remark 5.2.3) at the edge (soft and hard) is related to the limiting distribution by a derivative operation [52, 54, 56–58, 83, 110, 116]. The derivative of the RHS of (5.2.11) indeed yields the Gaussian factor in (5.2.13). Note that the factor $J_\beta(2\sqrt{a})^2$ appearing in (5.2.14) is also connected, via a derivative operation, to the hard-edge

limiting density of the singular values (5.2.6). This observation was made in [54]. In particular, we have the explicit identity

$$4\partial_a \left[a K_\beta^{\mathrm{Bes}}(4a, 4a) \right] = J_\beta \left(2\sqrt{a} \right)^2 . \tag{5.2.16}$$

The fact that the $1, k$-cross-covariance at the soft-hard edge is smaller by a factor $1/n$ than the limiting $1, k$-point function can be expected heuristically. Indeed, as the eigenradius does not share the same edge as the singular values in this case, Weyl's inequalities are much less felt and this translates in a vanishing local cross-covariance, in the large n limit.

5.3 Proof of Theorem 5.2.2

Let us start by defining

$$\mathbf{C}_n(r; a_1, a_2) := \int_0^\infty dt \int_0^r \frac{dv}{v} \varphi_n \left(\tfrac{v}{r}, t \right) K_n \left(a_1, -rt \right) \left[\delta(v - a_2) - K_n(v, a_2) \right] . \tag{5.3.1}$$

The main part of the proof of Theorem 5.2.2 relies on the following proposition that we shall prove first.

Proposition 5.3.1 *For the Jacobi ensemble with Pólya weight $w_{\mathrm{Jac}}(x) = x^\alpha (1-x)^{\beta+n-1} \Theta(1-x)$, $\alpha > -1, \beta > 0$ fixed, for all $(r, \lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}_+^2$, the limit*

$$\lim_{n\to\infty} \sqrt{n}\, \mathbf{C}_n \left(\frac{1}{2} - \frac{r}{\sqrt{n}}; 1 - \frac{\lambda_1}{n^2}, 1 - \frac{\lambda_2}{n^2} \right) = \frac{4}{\sqrt{\pi}} e^{-4r^2} \left(\frac{\lambda_1}{\lambda_2} \right)^{\beta/2} J_\beta \left(2\sqrt{\lambda_1} \right) J_\beta \left(2\sqrt{\lambda_2} \right) \tag{5.3.2}$$

holds pointwise.

Proof of Proposition 5.3.1. Explicitly,

$$\begin{aligned} \mathbf{C}_n(r; a_1, a_2) = &\Theta(r - a_2) \int_0^\infty dt \frac{1}{a_1} \varphi_n \left(\frac{a_1}{r}, t \right) K_n \left(a_1, -rt \right) \\ &- \int_0^\infty dt \int_0^r \frac{dv}{v} \varphi_n \left(\frac{v}{r}, t \right) K_n \left(a_1, -rt \right) K_n(v, a_2). \end{aligned} \tag{5.3.3}$$

From there, one first notices that, after setting $a_2 = 1 - \frac{\lambda_2}{n^2}$, the Heaviside function in (5.3.3) will vanish in the limit as for all fixed $(r, \lambda_2) \in \mathbb{R} \times \mathbb{R}_+$,

$$\lim_{n\to\infty} \Theta \left(\frac{r}{\sqrt{n}} + \frac{\lambda_2}{n^2} - \frac{1}{2} \right) = 0. \tag{5.3.4}$$

The term of interest is therefore

$$\int_0^r dv \int_0^\infty dt K_n\left(1-\tfrac{\lambda_1}{n^2}, -rt\right) K_n\left(v, 1-\tfrac{\lambda_2}{n^2}\right) \frac{\left(1-\frac{v}{r}\right)^{n-2}}{r^2(1+t)^{n+2}} \times \left[(1+rt)-(1-v)-\tfrac{1}{n}(r+vt)\right]. \tag{5.3.5}$$

Remark 5.3.2 *We keep r as it is, at the moment, because replacing it by $\frac{1}{2}-\frac{r}{\sqrt{n}}$ does not yield any simplification. We will proceed with the shift and scaling of r at the end, to keep expressions as simple as possible. However, in order not to run in unnecessary difficulty, one needs to take r in compact subsets of $]0,1[$ and can arbitrarily choose to take $r \in \left[\frac{1}{4}, \frac{3}{4}\right]$.*

As we focus on the first term of the asymptotic expansion of (5.3.5) when $n \to \infty$, it appears that one needs to compute the following integrals

$$\int_0^\infty dt\, K_n\left(1-\tfrac{\lambda_1}{n^2}, -rt\right)(1+t)^{-(n+2)}(1+rt)^\gamma, \quad \gamma = 0, 1 \tag{5.3.6}$$

and

$$\int_0^r dv\, K_n\left(v, 1-\tfrac{\lambda_2}{n^2}\right)\left(1-\frac{v}{r}\right)^{n-2}(1-v)^\gamma, \quad \gamma = 0, 1. \tag{5.3.7}$$

Unfortunately, none of the factors inside the integral admits an individual limit, the goal is therefore to get a good asymptotic of those integrals using Laplace's approximation method.

Asymptotic expansion of the kernel

The first step is to get an asymptotic expansion, as $n \to \infty$, of the integrands in (5.3.6) and (5.3.7), to be able to use Laplace's method. For this, we need the following propositions. From [134, p.196] we have the following asymptotic for Jacobi polynomials outside the orthogonality interval.

Proposition 5.3.3 *For all $x \in \mathbb{C} \setminus [0,1]$ and arbitrary fixed $\alpha, \beta \in \mathbb{R}$ an asymptotic expansion of $P_n^{(\alpha,\beta)}$ is*

$$P_n^{(\alpha,\beta)}(1-2x) = A_n^{(\alpha,\beta)}(x)\left[1+O\left(\frac{1}{n}\right)\right], \quad \text{as} \quad n \to \infty, \tag{5.3.8}$$

where

$$A_n^{(\alpha,\beta)}(x) := \frac{1}{2\sqrt{\pi n}} \frac{\left(1+\sqrt{\frac{x}{x-1}}\right)^{\alpha+\beta+n+1}}{\left(1-\sqrt{\frac{x}{x-1}}\right)^n \left(\frac{x}{x-1}\right)^{\frac{\alpha}{2}+\frac{1}{4}}}. \tag{5.3.9}$$

The expansion holds uniformly in the exterior of any closed contour enclosing $[0,1]$, in the sense the ratio goes to 1.

The Mehler–Heine formula [134] gives the following asymptotic for Jacobi polynomials near the right boundary of the orthogonality interval.

Proposition 5.3.4 *For any fixed $\alpha > -1, \beta > 0$, an asymptotic expansion of $P_n^{(\alpha,\beta)}$ is*

$$P_n^{(\alpha,\beta)}\left(\frac{2\lambda}{n^2} - 1\right) = B_n^{(\alpha,\beta)}(\lambda) + O\left(n^{\beta-1}\right), \quad \text{as} \quad n \to \infty, \tag{5.3.10}$$

where

$$B_n^{(\alpha,\beta)}(\lambda) := (-1)^n n^\beta \lambda^{-\frac{\beta}{2}} J_\beta\left(2\sqrt{\lambda}\right). \tag{5.3.11}$$

The expansion holds uniformly for λ in compact subsets of $\mathbb{R}_+$.

This leads to the following corollaries, where we retain the dependence of one of the kernel's arguments in the error term. Namely, the argument on which we integrate; cf. (5.3.6) and (5.3.7).

Corollary 5.3.5 *For any fixed , $\alpha > -1, \beta > 0$, the expansion*

$$K_n\left(1 - \frac{\lambda}{n^2}, -x\right) \underset{n\to\infty}{=} -n\left(\frac{\lambda}{n^2}\right)^\beta \left[B_n^{(\alpha,\beta)}(\lambda)\frac{P_n^{(\alpha,\beta-1)}(1+2x)}{1+x} + O\left(n^{\beta-1}\frac{\mathcal{E}_n(-x)}{(1+x)}\right)\right] \tag{5.3.12}$$

holds uniformly for all $x > 0$ and for λ in compact subsets of $\mathbb{R}_+$. $\mathcal{E}_n$ is the polynomial

$$\mathcal{E}_n(x) := P_n^{(\alpha,\beta-1)}(1-2x) + P_n^{(\alpha,\beta)}(1-2x) + P_{n-1}^{(\alpha,\beta)}(1-2x). \tag{5.3.13}$$

Remark 5.3.6 *Note that $B_n^{(\alpha,\beta)}(\lambda) \underset{n\to\infty}{=} O(n^\beta)$, therefore, the big O in (5.3.12) is really an error term.*

Proof of Corollary 5.3.5. Using the Christoffel-Darboux formula (5.2.1) and Proposition 5.3.4 we have,

$$\begin{aligned} K_n\left(1 - \frac{\lambda}{n^2}, -x\right) \underset{n\to\infty}{=} a_n\left(\frac{\lambda}{n^2}\right)^\beta \frac{1}{1+x}\Big[&\left(B_n^{(\alpha,\beta)}(\lambda) + O\left(n^{\beta-1}\right)\right) P_{n-1}^{(\alpha,\beta)}(1+2x) \\ &- \left(B_{n-1}^{(\alpha,\beta)}(\lambda) + O\left(n^{\beta-1}\right)\right) P_n^{(\alpha,\beta)}(1+2x)\Big]. \end{aligned} \tag{5.3.14}$$

Note that the respective domains of the polynomials do not intersect. Thus, there is no cancellation of the leading terms.

First, one has to notice that

$$B_{n-1}^{(\alpha,\beta)}(\lambda) \underset{n\to\infty}{=} -B_n^{(\alpha,\beta)}(\lambda)\left[1 + O\left(\frac{\beta}{n}\right)\right]. \tag{5.3.15}$$

Therefore, one can factorise by $B_n^{(\alpha,\beta)}(\lambda)$. Then, using the relation

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha,\beta+1)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x), \tag{5.3.16}$$

one gets

$$P_n^{(\alpha,\beta)}(1+2x) + P_{n-1}^{(\alpha,\beta)}(1+2x) \underset{n\to\infty}{=} 2P_n^{(\alpha,\beta-1)}(1+2x) + O\left(\frac{\mathcal{E}_n(-x)}{n}\right), \tag{5.3.17}$$

which yields

$$K_n\left(1-\frac{\lambda}{n^2}, -x\right) \underset{n\to\infty}{=} 2a_n\left(\frac{\lambda}{n^2}\right)^{\beta}\left[B_n^{(\alpha,\beta)}(\lambda)\frac{P_n^{(\alpha,\beta-1)}(1+2x)}{1+x} + O\left(n^{\beta-1}\frac{\mathcal{E}_n(-x)}{(1+x)}\right)\right]. \tag{5.3.18}$$

Finally, using the following asymptotic equivalent for a_n (5.2.3),

$$a_n \underset{n\to\infty}{\sim} -\frac{n}{2}, \tag{5.3.19}$$

finishes the proof. ■

Corollary 5.3.7 *For any fixed $\alpha > -1, \beta > 0$, the expansion*

$$K_n\left(v, 1-\frac{\lambda}{n^2}\right) \underset{n\to\infty}{=} -n\, v^{\alpha}(1-v)^{\beta-1}\left[B_n^{(\alpha,\beta)}(\lambda)P_n^{(\alpha,\beta-1)}(1-2v) + O\left(n^{\beta-1}\mathcal{E}_n(v)\right)\right]. \tag{5.3.20}$$

holds uniformly for λ in compact subsets of $\mathbb{R}_+$ and $v \in [0, 1-\epsilon]$, for any $\epsilon \in]0,1[$. The polynomial $\mathcal{E}_n$ is given by (5.3.13).

Proof of Corollary 5.3.7. In the same spirit as the proof of Corollary 5.3.5, we use the Christoffel-Darboux formula (5.2.1) and Proposition 5.3.4. This yields

$$\begin{aligned} K_n\left(v, 1-\frac{\lambda}{n^2}\right) \underset{n\to\infty}{=} a_n v^{\alpha}(1-v)^{\beta}\Big[&P_n^{(\alpha,\beta)}(1-2v)\left(B_{n-1}^{(\alpha,\beta)}(\lambda) + O\left(n^{\beta-1}\right)\right) \\ &- P_{n-1}^{(\alpha,\beta)}(1-2v)\left(B_n^{(\alpha,\beta)}(\lambda) + O\left(n^{\beta-1}\right)\right)\Big]. \end{aligned} \tag{5.3.21}$$

Here again, there is no cancellation of the leading terms as the arguments of the kernel are never equal to each other for n large enough, namely $n^2 > \lambda/\epsilon$ (cf. Remark 5.3.2). Following the same steps as in the proof of Corollary 5.3.5, the result follows. ■

From Corollary 5.3.5 and Corollary 5.3.7, one sees that the problem of getting the large n asymptotic of (5.3.5) can be broken down to getting the large n asymptotic of the following integrals

$$\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}(r) := \int_0^r dv\, v^{\alpha}(1-v)^{\beta-1+\gamma}P_n^{(\alpha,\beta-1)}(1-2v)\left(1-\frac{v}{r}\right)^{n-2}, \quad \gamma = 0, 1 \tag{5.3.22}$$

and

$$\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r) := \int_0^\infty dt(1+t)^{-(n+2)} \frac{P_n^{(\alpha,\beta-1)}(1+2rt)}{(1+rt)^{1-\gamma}}, \quad \gamma = 0,1. \tag{5.3.23}$$

Note that the error terms in (5.3.12) and (5.3.20) are composed of the same polynomials as the leading terms, with same degree. Indeed, $O(\mathcal{E}_n(x))$ is some linear combination of the Jacobi polynomials composing $\mathcal{E}_n$. Therefore, after integration, the error terms will respectively be expressed as some linear combination of $\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}$ and $\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}$, up to a shift in the parameters α, β, which are n-independent. The error terms will remain $1/n$ smaller than the leading terms and can, therefore, be neglected in the limit $n \to \infty$; see Remark 5.3.6.

Computation of $\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}(r)$

Let us focus first on $\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}(r)$, $\gamma = 0,1$, as one does not need to use the Laplace's method in this case. Indeed, one can use Rodrigues' formula for the Jacobi polynomials $P_n^{(\alpha,\beta)}$, which, under the mapping $x \mapsto 1-2x$, is given by

$$x^\alpha (1-x)^\beta P_n^{(\alpha,\beta)}(1-2x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^{\alpha+n}(1-x)^{\beta+n}\right]. \tag{5.3.24}$$

Then, integrating by part, respectively $n-2$ and $n-1$ times, yields

$$\mathcal{J}_{n,0}^{(\alpha,\beta)}(r) = \frac{r^{\alpha+1}(1-r)^{n+\beta-2}}{(n-1)}\left[1-2r+\frac{\alpha - r(\alpha+\beta-1)}{n}\right] \tag{5.3.25}$$

and

$$\mathcal{J}_{n,1}^{(\alpha,\beta)}(r) = \frac{r^{\alpha+1}(1-r)^{n+\beta-1}}{(n-1)}\left[1-r+\frac{\alpha - r(\alpha+\beta)}{n}\right]. \tag{5.3.26}$$

Asymptotic equivalent of $\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r)$

Let us now find asymptotic equivalents of $\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r)$, $\gamma = 0,1$. Once again the integral can be computed explicitly, however this yield a sum of hypergeometric functions for which an asymptotic equivalent is hard to get. Instead, we will use Laplace's approximation method. For this, we need the asymptotic for $P_n^{(\alpha,\beta-1)}$ given by Proposition 5.3.3. However, one needs to stay away from $t=0$, as the approximation is not uniform there. We then introduce a n-independent cut-off $t_\varepsilon > 0$ that we can conveniently express in terms of a small enough $\varepsilon > 0$ as

$$\varepsilon = \sqrt{\frac{rt_\varepsilon}{1+rt_\varepsilon}}. \tag{5.3.27}$$

We will see how small ε needs to be later on. To estimate the contribution of the integral

$$\int_0^{t_\varepsilon} dt(1+t)^{-(n+2)} \frac{P_n^{(\alpha,\beta-1)}(1+2rt)}{(1+rt)^{1-\gamma}}, \quad \gamma = 0,1, \tag{5.3.28}$$

one can bound $P_n^{(\alpha,\beta-1)}(1+2rt)$ by its maximum, which is attained at $t=t_\varepsilon$ and take the very rough upper bound

$$\int_0^{t_\varepsilon} dt(1+t)^{-(n+2)}\frac{P_n^{(\alpha,\beta-1)}(1+2rt)}{(1+rt)^{1-\gamma}} \leqslant \frac{1}{n+1}P_n^{(\alpha,\beta-1)}(1+2rt_\varepsilon), \quad \gamma=0,1. \tag{5.3.29}$$

To estimate the order of the upper bound as $n\to\infty$, one can use the approximation (5.3.8) as t_ε is bounded away from 0 and n-independent. This yields, in terms of ε,

$$\frac{1}{n+1}P_n^{(\alpha,\beta-1)}(1+2rt_\varepsilon) \underset{n\to\infty}{=} O\left(\frac{1}{n^{3/2}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n\right), \tag{5.3.30}$$

for r in compact subsets of $]0,1[$; see Remark 5.3.2. Using this estimate, we will see later that the integral (5.3.28) is negligible compared to its complement.

The plan is thus to apply Laplace's method on the following integral for $n\to\infty$,

$$\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r) := \int_{t_\varepsilon}^{\infty} dt(1+t)^{-(n+2)}\frac{A_n^{(\alpha,\beta-1)}(-rt)}{(1+rt)^{1-\gamma}}, \quad \gamma=0,1. \tag{5.3.31}$$

As the asymptotic expansion (5.3.8) is uniform on the integration domain, we have

$$\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} \mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r)\left[1+O\left(\frac{1}{n}\right)\right] + O\left(\frac{1}{n^{3/2}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n\right), \tag{5.3.32}$$

for all r in compact subsets of $]0,1[$, where both $\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r)$ and $\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r)$, are convergent integrals due to the factor $(1+t)^{-(n+2)}$.

We now prepare the integral $\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r)$ for Laplace's method. Proceeding with a change of variable $y=\sqrt{\frac{rt}{1+rt}}$, the change of measure is then $dt=\frac{2ydy}{r(1-y^2)^2}$ and the integral reads

$$\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r) = \frac{2}{r}\int_\varepsilon^1 dy\, y(1-y^2)^{n+1-\gamma}\left(1+y^2(\tfrac{1}{r}-1)\right)^{-(n+2)} A_n^{(\alpha,\beta-1)}\left(-\frac{y^2}{1-y^2}\right). \tag{5.3.33}$$

One can then rewrite the integrand

$$\begin{aligned}\frac{g_{r,\gamma}(y)}{\sqrt{n}}\exp(-nf_r(y)) &= \frac{2}{r}y(1-y^2)^{n+1-\gamma}\left(1+y^2(\tfrac{1}{r}-1)\right)^{-(n+2)} A_n^{(\alpha,\beta-1)}\left(-\frac{y^2}{1-y^2}\right)\\ &= \frac{1}{r\sqrt{n\pi}}y^{\frac{1}{2}-\alpha}(1-y)^{1-\gamma}\frac{(1+y)^{2n+\alpha+\beta+1-\gamma}}{\left(1+y^2(\frac{1}{r}-1)\right)^{n+2}},\end{aligned} \tag{5.3.34}$$

defining, as follows, the functions f_r and $g_{r,\gamma}$, for which we already give the explicit

expressions of the needed derivatives

$$f_r(y) := \ln\big(1+y^2(\tfrac{1}{r}-1)\big) - 2\ln(1+y),$$

$$f_r'(y) = 2\frac{y(1-r)-r}{(1+y)(r+y^2(1-r))}, \tag{5.3.35}$$

$$f_r''(y) = 2\frac{r(1+y)(1+y+4y^2) - 2r^2y(1+y)^2 - y^2(1+2y)}{(1+y)^2(r+y^2(1-r))^2}$$

and

$$g_{r,\gamma}(y) := \frac{1}{r\sqrt{\pi}} y^{\frac{1}{2}-\alpha}(1-y)^{1-\gamma}\frac{(1+y)^{\alpha+\beta+1-\gamma}}{\big(1+y^2(\frac{1}{r}-1)\big)^2},$$

$$\begin{aligned} g_{r,\gamma}'(y) =& \frac{y^{-\alpha-\frac{1}{2}}(1+y)^{\alpha+\beta}(1-y^2)^{-\gamma}}{2r^2\sqrt{\pi}\left(1+\left(\frac{1}{r}-1\right)y^2\right)^3}\Big[y^2\left(-2\alpha+y^2(-2\beta+4\gamma+3)+2y(\alpha+\beta)-7\right) \\ &+ r\left(y^2-1\right)\left(2\alpha+y^2(2\beta-4\gamma-3)-2y(\alpha+\beta)-1\right)\Big]. \end{aligned} \tag{5.3.36}$$

The function f_r admits a unique critical point; corresponding to a minimum located at $y = \frac{r}{1-r}$. For $r > \frac{1}{2}$, the critical point escapes the domain of integration, the minimum is then on the boundary at $y = 1$. However, $g_{r,0}(1) = 0$, hence the need to expand $g_{r,\gamma}$ to first order and f_r to second order. Note that, despite the term $(1-y^2)^{-1}$, $g_{r,1}'$ does not have a singularity at $y = 1$, as it is removable. Denoting,

$$\begin{aligned} \widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) :=& \int_\varepsilon^1 \frac{dy}{\sqrt{n}}\left[g_{r,\gamma}(y_r) + (y-y_r)g_{r,\gamma}'(y_r)\right] \\ &\times \exp\left(-n\left[f_r(y_r) + (y-y_r)f_r'(y_r) + \frac{(y-y_r)^2}{2}f_r''(y_r)\right]\right), \quad \gamma = 0,1, \end{aligned} \tag{5.3.37}$$

with

$$y_r = \begin{cases} \dfrac{r}{1-r}, & 0 < r \leqslant \dfrac{1}{2} \\ 1, & \dfrac{1}{2} < r < 1 \end{cases}. \tag{5.3.38}$$

We thus have

$$\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} \widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r)\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right], \tag{5.3.39}$$

uniformly for r in compact subsets of $]0,1[$. Note that the relative error is, here, of order $n^{-1/2}$ and not n^{-1} due to $g_{r,0}$ vanishing at $y = 1$; see [139, p.37]. To compute explicitly $\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r)$, one proceeds with the change of variable $u = y_r - y$.

The expression becomes

$$\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) = \int_{y_r-1}^{y_r-\varepsilon} \frac{du}{\sqrt{n}} \left[g_{r,\gamma}(y_r) - u g'_{r,\gamma}(y_r)\right] \exp\left(-n\left[f_r(y_r) - u f'_r(y_r) + \frac{u^2}{2} f''_r(y_r)\right]\right) \tag{5.3.40}$$

and can be computed using the following lemma, which is obtained with a simple integration by part.

Lemma 5.3.8 *Let $a, b, c, d, h, x_0, x_1 \in \mathbb{R}$, $a \neq 0$,*

$$\begin{aligned}&\int_{x_0}^{x_1} (d - hx) \exp\left(-\tfrac{a}{2}x^2 - bx + c\right) dx \\ &= \tfrac{h}{a}\left(e^{-\frac{ax_1^2}{2} - bx_1 + c} - e^{-\frac{ax_0^2}{2} - bx_0 + c}\right) + \sqrt{\tfrac{\pi}{2}} e^{\frac{b^2}{2a} + c} \tfrac{(ad+bh)}{a^{3/2}} \left(\operatorname{erf}\left(\tfrac{ax_1+b}{\sqrt{2a}}\right) - \operatorname{erf}\left(\tfrac{ax_0+b}{\sqrt{2a}}\right)\right),\end{aligned} \tag{5.3.41}$$

with

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t, \quad x \in \mathbb{R}. \tag{5.3.42}$$

Using Lemma 5.3.8 and the following expressions

$$f'_r(1) = (1-2r), \quad f''_r(1) = -\left(4r^2 - 6r + \frac{3}{2}\right), \quad f''_r\left(\frac{r}{1-r}\right) = 2\frac{(1-r)^3}{r} \tag{5.3.43}$$

and

$$g'_{r,\gamma}\left(\frac{r}{1-r}\right) = -\frac{4r^2(\alpha+\beta-\gamma-3) - r(6\alpha+2\beta-10) + 2\alpha - 1}{2\sqrt{\pi}\, r^{\alpha+\frac{3}{2}} (1-r)^{\beta-\frac{1}{2}}} \left(\frac{(1-r)^2}{1-2r}\right)^{\gamma}, \tag{5.3.44}$$

$$g'_{r,\gamma}(1) = -\frac{2^{\alpha+\beta-1}}{\sqrt{\pi}} r\Big(4 + \gamma(\alpha - \beta + 3 - 8r)\Big), \tag{5.3.45}$$

one gets, after simplifications, for $r \leqslant \frac{1}{2}$,

$$\begin{aligned}\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} &\frac{(1-2r)^{-\gamma}(1-r)^{2\gamma}}{4n\sqrt{\pi}\, r^{\alpha+1}(1-r)^{\beta+n+2}} \Bigg[2\sqrt{\pi}(1-2r)r \\ &\times \left[\operatorname{erf}\left(\sqrt{n}(1-2r)\sqrt{\tfrac{1-r}{r}}\right) + \operatorname{erf}\left(\sqrt{n}(r-(1-r)\epsilon)\sqrt{\tfrac{1-r}{r}}\right)\right] \\ &+ \frac{1}{\sqrt{n}}\sqrt{\tfrac{r}{1-r}}\left(e^{-n(1-2r)^2\frac{(1-r)}{r}} - e^{-n(r-(1-r)\epsilon)^2\frac{(1-r)}{r}}\right) \\ &\times \left(4r^2(\alpha+\beta-\gamma-3) - 2r(3\alpha+\beta-5) + 2\alpha - 1\right)\Bigg]\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\end{aligned} \tag{5.3.46}$$

and, for $r > \frac{1}{2}$,

$$\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} \frac{2^{2n+\alpha+\beta+2(1-\gamma)} r^{n+1} (\alpha-\beta+7-8r)^{\gamma}}{n\sqrt{\pi}\,(4(3-2r)r-3)^{\frac{3}{2}}} \left[\frac{1}{\sqrt{n}} \sqrt{4(3-2r)r-3} + \sqrt{\pi} \left[2r-1-\gamma\left(\frac{4(3-2r)r-3}{\alpha-\beta+7-8r}\right)\right] \exp\left(\frac{n(2r-1)^2}{4(3-2r)r-3}\right) \times \left(\operatorname{erf}\left(\frac{\sqrt{n}(2r-1)}{\sqrt{4(3-2r)r-3}}\right) - 1 \right) \right] \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right). \tag{5.3.47}$$

Note that the ε dependence completely drops out here. This is also the case in (5.3.46) if one chooses $\varepsilon < \frac{r}{1-r}$. As r is bounded away from 0, this is certainly always possible; see Remark 5.3.2. Actually, coming back to the remainder integral (5.3.28), one needs to take ε slightly smaller, to keep its relative contribution negligible. Indeed, taking $\varepsilon < \frac{r}{1-r}$, for r in compact subsets of $]0,1[$,

$$\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} O\left(\frac{1}{n(1-r)^n}\right), \quad r \leqslant 1/2, \tag{5.3.48}$$

and

$$\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} O\left(\frac{(4r)^n}{n}\right), \quad r > 1/2. \tag{5.3.49}$$

The latter is obtained using the asymptotic expansion of the complementary error function

$$1 - \operatorname{erf}(x) \underset{x\to\infty}{=} \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 - \frac{1}{2x^2} + O\left(\frac{1}{x^4}\right)\right]. \tag{5.3.50}$$

Therefore, comparing the order of $\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r)$ with the order of the remainder integral (5.3.30), one needs to choose ε such that, for r on compact subsets of $]0,1[$,

$$\begin{cases} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)(1-r) < 1, & r \leqslant 1/2 \\ \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\frac{1}{4r} < 1, & r > 1/2 \end{cases}, \tag{5.3.51}$$

which can be reduced to

$$\varepsilon < \min\left\{ \frac{r}{2-r}, \frac{1}{3} \right\}. \tag{5.3.52}$$

This choice of ε guarantees that the remainder integral (5.3.28) is exponentially small compared to $\mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r)$ as $n \to \infty$. Consequently, the expansion (5.3.32) reduces to

$$\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r) \underset{n\to\infty}{=} \mathcal{Y}_{n,\gamma}^{(\alpha,\beta)}(r) \left[1 + O\left(\frac{1}{n}\right)\right] \underset{n\to\infty}{=} \widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r) \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]. \tag{5.3.53}$$

Asymptotic equivalent of the $1,k$-cross-covariance function

With all the asymptotic expansions found above, we are in good position to find the desired asymptotic equivalent of the $1,k$-cross-covariance density. Let us first

notice that

$$\int_0^r dv\, v^{\alpha+1}(1-v)^{\beta-1}P_n^{(\alpha,\beta-1)}(1-2v)\left(1-\frac{v}{r}\right)^{n-2} = \mathcal{J}_{n,0}^{(\alpha,\beta)}(r) - \mathcal{J}_{n,1}^{(\alpha,\beta)}(r) \quad (5.3.54)$$

and

$$\int_0^\infty dt(1+t)^{-(n+2)}\frac{P_n^{(\alpha,\beta-1)}(1+2rt)}{1+rt}t = \frac{\mathcal{I}_{n,1}^{(\alpha,\beta)}(r) - \mathcal{I}_{n,0}^{(\alpha,\beta)}(r)}{r}. \quad (5.3.55)$$

Then, gathering the asymptotic expansions (5.3.12) and (5.3.20), along with the integrals $\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}(r)$ (5.3.22) and $\mathcal{I}_{n,\gamma}^{(\alpha,\beta)}(r)$ (5.3.23), one gets the following asymptotic equivalent

$$\begin{aligned}&\int_0^r dv\int_0^\infty dtK_n\left(1-\tfrac{\lambda_1}{n^2},-rt\right)K_n(v,1-\tfrac{\lambda_2}{n^2})\left(1-\frac{v}{r}\right)^{n-2}(1+t)^{-(n+2)}\\ &\times\frac{1}{r^2}\left[(1+rt)-(1-v)-\tfrac{1}{n}(r+vt)\right]\\ &\underset{n\to\infty}{\sim} n^{2-2\beta}\lambda_1^\beta B_n^{(\alpha,\beta)}(\lambda_1)B_n^{(\alpha,\beta)}(\lambda_2)\mathcal{G}_n^{(\alpha,\beta)}(r)\end{aligned} \quad (5.3.56)$$

with

$$\begin{aligned}\mathcal{G}_n^{(\alpha,\beta)}(r) :=& \frac{1}{r^2}\Bigg(\mathcal{I}_{n,0}^{(\alpha,\beta)}(r)\left[\left(\frac{1}{nr}-\frac{r}{n}\right)\mathcal{J}_{n,0}^{(\alpha,\beta)}(r)-\left(1+\frac{1}{nr}\right)\mathcal{J}_{n,1}^{(\alpha,\beta)}(r)\right]\\ &+\mathcal{I}_{n,1}^{(\alpha,\beta)}(r)\left[\left(1-\frac{1}{nr}\right)\mathcal{J}_{n,0}^{(\alpha,\beta)}(r)+\frac{1}{nr}\mathcal{J}_{n,1}^{(\alpha,\beta)}(r)\right]\Bigg).\end{aligned} \quad (5.3.57)$$

The explicit expression of $\mathcal{G}_n^{(\alpha,\beta)}(r)$ is rather cumbersome, however, there is some simplification by noting

$$\begin{aligned}&\left(\frac{1}{nr}-\frac{r}{n}\right)\mathcal{J}_{n,0}^{(\alpha,\beta)}(r)-\left(1+\frac{1}{nr}\right)\mathcal{J}_{n,1}^{(\alpha,\beta)}(r)\\ =&-\frac{r^{\alpha+1}(1-r)^{\beta+n-1}}{n}\left(1-r+\frac{1+\alpha-r(\alpha+\beta-1)}{n}\right)\end{aligned} \quad (5.3.58)$$

and

$$\begin{aligned}&\left(1-\frac{1}{nr}\right)\mathcal{J}_{n,0}^{(\alpha,\beta)}(r)+\frac{1}{nr}\mathcal{J}_{n,1}^{(\alpha,\beta)}(r)\\ =&\frac{r^{\alpha+1}(1-r)^{\beta+n-2}}{n}\left(1-2r+\frac{1+\alpha-r(\alpha+\beta)}{n}\right).\end{aligned} \quad (5.3.59)$$

The asymptotic equivalent $\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r)$ of the integrals $\mathcal{J}_{n,\gamma}^{(\alpha,\beta)}(r)$ are, at the moment, defined piecewise, depending on the value of r. Both pieces must agree at $r=1/2$, which is precisely the value around which we are zooming.

For $r \leqslant 1/2$, using the asymptotic equivalent $\widehat{\mathcal{Y}}_{n,\gamma}^{(\alpha,\beta)}(r)$ and proceeding with sim-

plifications, one gets the following asymptotic equivalent

$$\mathcal{G}_n^{(\alpha,\beta)}(r) \underset{n\to\infty}{\sim} \frac{-1}{n^2(1-r)^3\sqrt{\pi}}\left[\frac{1}{\sqrt{n}}\left(\sqrt{r(1-r)}-\frac{1}{n}\frac{P_0(r)}{4\sqrt{r(1-r)}}\right)e^{-n\frac{(1-r)}{r}(1-2r)^2}\right.$$
$$\left.+\frac{\sqrt{\pi}}{2n}\left[1+\operatorname{erf}\left((1-2r)\sqrt{\frac{n(1-r)}{r}}\right)\right](r(\alpha+\beta-2)-\alpha)\right] \tag{5.3.60}$$

with the polynomial

$$\begin{aligned}P_0(r) :=& 2r^2\left[\alpha^2+2\alpha(\beta-2)+(\beta-4)\beta+6\right]\\ &+r\left[\alpha(-4\alpha-4\beta+9)+\beta-6\right]+\alpha(2\alpha-1).\end{aligned} \tag{5.3.61}$$

From there, one can proceed with the change of variable $r \mapsto \frac{1}{2}-\frac{r}{\sqrt{n}}$. This gives

$$\mathcal{G}_n^{(\alpha,\beta)}\left(\frac{1}{2}-\frac{r}{\sqrt{n}}\right) \underset{n\to\infty}{=} -\frac{4e^{-4r^2}}{\sqrt{\pi}n^{5/2}}+O\left(\frac{1}{n^3}\right). \tag{5.3.62}$$

For $r > 1/2$, one gets the following asymptotic equivalent

$$\begin{aligned}\mathcal{G}_n^{(\alpha,\beta)}(r) \underset{n\to\infty}{\sim} &-\frac{2^{\alpha+\beta+2n}r^{\alpha+n}(1-r)^{\beta+n-2}}{n^{5/2}\left(8r^2-12r+3\right)^2\sqrt{\pi}}\left[\left(8r^2-12r+3\right)\left(P_1(r)+\frac{1}{n}P_2(r)\right)\right.\\ &-\sqrt{\pi}\sqrt{-n\left(8r^2-12r+3\right)}\left((1-2r)^2(-\alpha+\beta+2r)+\frac{1}{n}P_3(r)\right)\\ &\left.\times e^{-\frac{n(1-2r)^2}{8r^2-12r+3}}\left(\operatorname{erf}\left(\frac{n(2r-1)}{\sqrt{-n\left(8r^2-12r+3\right)}}\right)-1\right)\right],\end{aligned} \tag{5.3.63}$$

with

$$\begin{aligned}P_1(r) :=& 12r^2-2r(\alpha-\beta+7)+\alpha-\beta+3,\\ P_2(r) :=& 4r^2(\alpha+\beta+1)-r\left(\alpha^2+7\alpha-\beta^2+3\beta+8\right)+(\alpha+1)(\alpha-\beta+3),\\ P_3(r) :=& 8r^3+2r^2\left(-\alpha^2+\alpha+\beta^2+\beta-6\right)\\ &+r((\alpha-\beta)(3\alpha+\beta)-2\beta+2)-(\alpha+1)(\alpha-\beta).\end{aligned} \tag{5.3.64}$$

Then, using the asymptotic expansion of the complementary error function (5.3.50) and the same change of variable $r \mapsto \frac{1}{2}-\frac{r}{\sqrt{n}}$, one arrives at the following asymptotic equivalent

$$\begin{aligned}\mathcal{G}_n^{(\alpha,\beta)}\left(\frac{1}{2}-\frac{r}{\sqrt{n}}\right) \underset{n\to\infty}{\sim} &-\frac{4}{\sqrt{\pi}n^{5/2}}\left(1+\frac{2r}{\sqrt{n}}\right)^n\left(1-\frac{2r}{\sqrt{n}}\right)^n\left[1+\frac{1}{n}P_2(1/2)\right.\\ &\left.-\frac{1}{\sqrt{n}}\sqrt{\pi}e^{4r^2}\left(\operatorname{erf}(2r)-1\right)\right].\end{aligned} \tag{5.3.65}$$

The last step is to use the expansion

$$\left(1-\frac{4r^2}{n}\right)^n \underset{n\to\infty}{=} e^{-4r^2}\left(1+O\left(\frac{1}{n}\right)\right) \tag{5.3.66}$$

to arrive once again at

$$\mathcal{G}_n^{(\alpha,\beta)}\left(\frac{1}{2}-\frac{r}{\sqrt{n}}\right) \underset{n\to\infty}{=} -\frac{4e^{-4r^2}}{\sqrt{\pi}n^{5/2}}+O\left(\frac{1}{n^3}\right). \tag{5.3.67}$$

Plugging this back in (5.3.56) and then in (5.3.3) finishes the proof.

■

Proof of Theorem 5.2.2. Starting from Theorem 3.3.7 and following the same arguments as the proof of Corollary 4.2.7, one gets

$$\mathrm{cov}_{1,k}(r;a_1,\dots,a_k) = \frac{(n-k)!}{(n-1)!}\sum_{l,m=1}^{k}(-1)^{l+m}\det[K_n(a_b,a_c)]_{\substack{b\in[\![1,k]\!]\setminus\{l\}\\ c\in[\![1,k]\!]\setminus\{m\}}}\mathbf{C}_n(r;a_l,a_m). \tag{5.3.68}$$

Proceeding with the change of variables $r\to\frac{1}{2}-\frac{r}{\sqrt{n}}$ and $a_j\to 1-\frac{a_j}{n^2}$, $j=1,\dots,k$, and scaling by a factor n^{k+2}, one gets

$$\begin{aligned}&\frac{n^{k+2}}{n^{2k+1/2}}\mathrm{cov}_{1,k}\left(\frac{1}{2}-\frac{r}{\sqrt{n}};1-\frac{a_1}{n^2},\dots,1-\frac{a_k}{n^2}\right)\\ &=\frac{(n-k)!\,n^k}{n!}\sum_{l,m=1}^{k}\sqrt{n}\mathbf{C}_n\left(\frac{1}{2}-\frac{r}{\sqrt{n}};1-\frac{a_l}{n^2},1-\frac{a_m}{n^2}\right)\\ &\qquad\times(-1)^{l+m}\det\left[\frac{1}{n^2}K_n\left(1-\frac{a_b}{n^2},1-\frac{a_c}{n^2}\right)\right]_{\substack{b\in[\![1,k]\!]\setminus\{l\}\\ c\in[\![1,k]\!]\setminus\{m\}}}.\end{aligned} \tag{5.3.69}$$

Then, taking the limit $n\to\infty$ with the help of (4.6.25), the limiting kernel (5.2.5) and Proposition 5.3.1, one gets, using Jacobi's formula (A.2.4), that $\mathrm{cov}^{(\infty)}_{1,k\,\mathrm{SH}}(r;a_1,\dots,a_k)$ is equal to

$$\frac{4}{\sqrt{\pi}}e^{-4r^2}\partial_\mu\det\left[\left(\frac{a_b}{a_c}\right)^{\beta/2}4K_\beta^{\mathrm{Bes}}(4a_b,4a_c)+\mu\left(\frac{a_b}{a_c}\right)^{\beta/2}J_\beta(2\sqrt{a_b})J_\beta(2\sqrt{a_c})\right]_{b,c=1}^{k}\Bigg|_{\mu=0}. \tag{5.3.70}$$

The factor $\left(\frac{a_b}{a_c}\right)^{\beta/2}$ can be eliminated by elementary transformations on the determinant. Alternatively, as the dependence of a_b, a_c factorizes in the scaling limit of $\mathbf{C}_n$, (5.3.70) can also be expressed in the form

$$-\frac{4^k}{\sqrt{\pi}}e^{-4r^2}\det\begin{pmatrix}0 & J_\beta(2\sqrt{a_c})\\ J_\beta(2\sqrt{a_b}) & K_\beta^{\mathrm{Bes}}(4a_b,4a_c)\end{pmatrix}_{b,c=1}^{k}. \tag{5.3.71}$$

■

Chapter 6

Summary and outlook

Science is the belief in the ignorance of experts.

Richard P. Feynman
(1918–1988)

In this thesis we have studied the (collective[1]) relation between singular values and eigenvalues of $n \times n$ complex random matrices, both at global and local scales. As stressed in the introduction, the knowledge of both kinds of values is important in many applications. Indeed, eigenvalues and singular values do not necessarily capture the same information of the system—modelled by the complex matrix. Albeit being related by inequalities, there is no one-to-one correspondence between individual eigenvalues and individual singular values. Besides, as the eigenvalues are complex for general non-Hermitian matrices, each eigenvalue is described by 2 real numbers—its real and imaginary part in Cartesian coordinates or its radius and angle in polar coordinates—the inequalities in question hold for the modulus of the eigenvalues: the eigenradii.

On a heuristic level, it is therefore clear that one cannot find a bijection between the two sets of values as eigenvalues belong to a $2n$-dimensional space $\mathbb{C}^n$ while the singular values belong to a n-dimensional space $\mathbb{R}_+^n$. However, for systems where there is no information encoded in the eigenangles, a potential bijection can be found between the set of eigenradii which belong to $\mathbb{R}_+^n$ and the singular values. What we mean by no information being encoded in the eigenangles is that they follow a distribution independent of the distribution of the random matrix (see discussion below Theorem 2.3.5).

A consequence is that the eigenspectrum of such a complex random matrix is radially symmetric. Yet this is not enough to get a bijection between[2] singular values

[1] We did not study the individual relation between specific eigenvalues and singular values but only the relation between the two full collections of those values.

[2] With the help of a diagonal Haar distributed unitary matrix, one can build two different ensembles—non-bi-unitarily invariant—whose underlying distribution on the singular values will be the same, thus violating injectivity.

and eigenradii on the level of probability distributions. A sufficient condition is, however, for the probability distribution of complex random matrix to be bi-unitarily invariant. This means that the singular vectors[3] are uniformly (Haar) distributed on $\mathrm{U}(n)$. A direct consequence is that the distribution of the complex random matrix is entirely characterized by the underlying distribution of its singular values or, equivalently, its eigenradii. The difference being that while any distribution on the singular values can be traced back to a bi-unitarily invariant random matrix, it is not possible for all distributions on the eigenradii. This is reminiscent of the fact asking for bi-unitary invariance is stronger than simply conjugacy unitary invariance; cf. subsection 2.1.2.

The finding of such a bijection between the singular values and eigenradii distributions of bi-unitarily invariant matrices is the result of Kieburg and Kösters [91]. This can be seen as the finite dimensional analogue of Haagerup-Larsen theorem [19, 67] which gives the map between eigenvalues and singulars values of infinite dimensional version of bi-unitarily invariant operators, as well as the support of the different spectrum. Bridging the two comes Feinberg-Zee single ring theorem [48, 66, 128], which confirms that the distributions of the eigenvalues and singular values converge indeed, in the large n limit, to the corresponding spectral distributions of the infinite dimensional operator.

In this thesis, we have introduced the notion of j,k-point correlation functions/measures (Definition 2.3.7) which naturally extends the notion of k-point correlation functions[4] in the case of two different families of random variables. Then, to extract the correlations between the two families—eigenradii and singular values—we defined the j,k-cross-covariance density function (Definition 2.3.11) by subtracting the product of the j-point correlation functions on the eigenradii and the k-point function on the singular values to the j,k-point correlation function. This gives a measurement of the distance from statistical independence: a covariance measurement. It is worth noting that the n,n-point correlation measure lacks a density, owing to the deterministic relation between the product of the eigenradii and the product of the singular values, as expressed in (1.1.6). However, provided that the joint distribution of the singular values admits a density, all other j,k-point correlation measures do as well. Another way to extract the correlations between eigenradii and singular values would be to generalize the notion of cluster functions—also called truncated correlation functions; cf. [42, Eq.(5)] and [51, Eq.(5.3)]. While we did not looked at cluster functions in this work, they are—as the $1,k$-cross-covariance—defined as linear combinations of the different j,k-point functions.

In a first part, we considered n fixed. For $n=1$ and $n=2$ we found explicitly all the j,k-point measures, for any bi-unitarily invariant ensemble (Proposition 3.2.3, Proposition 3.2.4). For $n>2$, exploiting the bijection of [91], we were successful in finding a general formula—yet not very explicit—for the $1,k$-point function Theorem 3.3.1 and the conditional probability density of one eigenradius under fixed, pair-wise distinct, singular values (Corollary 3.3.2).

[3] The matrices U,V in (1.1.1).
[4] Normalized to probability density.

When assuming the underlying ensemble on the singular values is a polynomial ensemble, i.e. a particular kind of determinantal point process where the kernel is polynomial in one entry[5], the general formula for the $1, k$-point function becomes more explicit and involves only the kernel of the point process and some simple universal function (Theorem 3.3.7). A by-product has been to find a general formula for the 1-point function—level density function—on the eigenradii, which was not known for general bi-unitarily invariant ensembles (3.3.6) and not even for polynomial ensembles (3.3.13).

We also looked at a subclass of polynomial ensembles, called Pólya ensembles (of multiplicative type). For this subclass, all the formulas simplify further and depend only on a single function of one variable: the Pólya weight function. Furthermore, all the integrals involved in the formulas for polynomial ensembles can be carried out explicitly in the Pólya case. The subclass of Pólya ensembles is very interesting as they exhibit a lot of structure and enjoy some closure property; cf. (2.1.31). Moreover, it turns out that most of the classical complex random matrix ensembles in random matrix theory do have an underlying density on the singular values which follow a Pólya ensemble: Laguerre ensemble (1.1.12) (induced Ginibre ensemble), Jacobi ensemble (1.1.13) (ensemble of truncated unitary matrices), Cauchy-Lorentz ensemble (1.1.14), Muttalib-Borodin ensemble (4.4.1). This is not surprising, as more structured objects were studied first and are now considered classical.

In a second part, we studied the large n limit of the results obtained in the first part. Explicitly, we zoomed around the origin—which corresponds to a hard edge, where we expect non-trivial correlations between eigenradii and singular values to survive the large n limit, due to the deterministic Weyl's inequalities (1.1.8), (1.1.9) and (1.1.10). We found explicit formulas for the double scaling limit of the $1, k$-point function around the origin—and thus for the $1, k$-cross-covariance, both for polynomial ensembles (Theorem 4.2.5) and Pólya ensembles (Corollary 4.3.15). Without surprise, the formulas involve, respectively, the double scaling limit of the kernel and the double scaling limit of the Pólya weight. An unexpected finding was that the scaling, around the origin, of the smallest eigenradii is $\sqrt{n}$ larger[6] than the scaling of the smallest singular values. While this is clear from an analytical perspective, a general heuristic explanation remains elusive. We emphasize that while we do not exhibit new ensembles in this work, the result for Pólya ensembles remain true under composition (see Remark 4.3.5). Thus, one is free to compose finitely many different Pólya ensembles—verifying the hard edge assumptions—to create new ensembles and the results still hold.

We then looked at the particular case of the Jacobi ensemble (1.1.13), which admits a second hard edge around the upper edge of the support of the singular values, namely around 1. The upper edge for the eigenradius is in this case around $1/2$ and not 1. Moreover, the edge is soft for the eigenradius. The fact that the eigenradius does not share the same edge as the singular values, there, leads to vanishing correlations and thus to statistical independence. Mathematically, this translates into the $1, k$-cross-covariance function being of order $1/n$ smaller than the $1, k$-point

[5] In our case in the second entry.

[6] For squared eigenradii and singular values it becomes n.

function. The $1,k$-cross-covariance function can then be interpreted as the first leading correction term in the large n expansion of the $1,k$-point function around this hard-soft edge. The explicit structure of the limiting $1,k$-cross-covariance function can be put in relation with what has been observed in β-ensembles, i.e. that the leading term correction of the limiting density around an edge is related by a derivative operation to the limiting density.

Some problems remain open. First, it would be desirable to find a general formula for all the j,k-point functions, at least for polynomial ensembles or Pólya ensembles. This is challenging and the current strategy might not give a way to proceed beyond the $2,k$-point function. Precisely, the difficulty lies in computing the following quantity, with $\mathcal{C}(n) = \bigtimes_{k=1}^{n}(k + i\mathbb{R})$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathcal{C}(n)}\left[\prod_{k=1}^{n}\frac{ds_k}{2\pi i}\zeta(\varepsilon\,\mathrm{Im}\{s_k\})\right]\mathrm{perm}[r_b^{-s_c}]_{b,c=1}^{n}\frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}\\ &=\mathcal{M}_S^{-1}\left[\frac{\det\left[a_b^{s_c-1}\right]_{b,c=1}^{n}}{\Delta_n(s)\Delta_n(a)}\right](r),\end{aligned}\tag{6.0.1}$$

where the equality has to be understood in the sense of distributions. Another strategy might be to generalize the bijection of [91] on the level of distributions, or undo the group integrals which result in the formulation of (2.3.15). Let us also remind here that the strategy of direct integration and identification—essentially using Definition 2.3.7 and Lemma 3.2.1—used in the cases $n = 1$ and $n = 2$, becomes intractable for $n > 2$. Indeed, one needs to treat all possible arrangements individually—the number of which grows as $(n!)^2$.

Second, it would be interesting to relax some assumptions made both on the kernel and the Pólya weights for the large n asymptotic study presented in Chapter 4, as they are—for some of them—difficult to check. This implies a deeper understanding of the structure of polynomial and Pólya ensembles. For the former, it does not look promising as they lack structure. A priori, the only notable structural property that polynomial ensembles possess is inverse closure, inherited from the bi-unitarily invariant density on $\mathrm{GL}(n,\mathbb{C})$ (see Remark 2.1.16). Apart from the requirement that the determinant of the weight matrix be positive, no further restrictions are imposed, and there exists no general formula for constructing explicit biorthonormal families $\{p_j\}_{j=1}^{n}$ and $\{q_j\}_{j=1}^{n}$ used in the formulation of the kernel of the associated determinantal point process.

For Pólya ensembles, certain structural aspects remain to be understood. For instance, it seems the sequence of moments $\{m_p\}_{p\in\mathbb{N}}$ of the rescaled Pólya weight $\mathbf{w}_n(x) := \frac{\xi_n}{\nu_n}w_n\left(\frac{x}{\nu_n}\right)$—which corresponds to the Mellin transform of the weight $\mathbf{w}_n$ evaluated at positive integers—grows super-exponentially with respect to p. It is at least the case for all known examples (see Example 4.3.4 and (4.4.6) with Remark 4.3.5). This is likely due to some properties of Pólya frequency functions and the log-convexity of the Mellin transform on the positive real line.

Other questions relative to Pólya ensembles are still open: are Pólya ensembles

the only ensembles which are both determinantal on the singular values and the eigenvalues? If not, are they the unique ones that are both determinantal and enjoy the closure properties? We know they are not the only one to enjoy the closure property. Indeed, the composition of a polynomial ensemble and a Pólya ensemble is still a polynomial ensemble whose probability distribution is also explicitly given; see [60, 92]. Another interesting question would be to determine whether or not Pólya ensembles are dense in the set of polynomial ensembles.

Let us stress there has been some recent development related to Pólya ensembles. In [94], the notion of cyclic Pólya ensembles has been introduced to study distributions of eigenvalues of unitary matrices and the recent work of Wolfs [149] aimed at characterizing the intersection between Pólya ensemble—both of multiplicative and additive types—and orthogonal ensembles. This latter characterization, however, is quite limited, as it includes only two ensembles—Laguerre and Jacobi—while already excluding the Cauchy-Lorentz ensemble, which is one of the three classical ensembles known to lie in the intersection. Thus, some further development in the exploration of the Pólya class are definitely to expect. Stay tuned.

Appendix A

Supplementary material

I have had my results for a long time: but I do not yet know how I am to arrive at them.

CARL FRIEDRICH GAUSS
(1777–1855)

A.1 Proof of Proposition 4.3.12

Proof of Proposition 4.3.12. The result dates back at least to the introduction of the Bessel kernel in [143] where only the idea of the proof is given. Let us start from the integral representation and do the change of variable $\sqrt{\tau} = t$

$$K_\alpha^{\mathrm{Bes}}(a^2, b^2) = \frac{1}{4}\int_0^1 d\tau J_\alpha(a\sqrt{\tau})J_\alpha(b\sqrt{\tau}) = \frac{1}{2}\int_0^1 dt\, t\, J_\alpha(at)J_\alpha(bt). \tag{A.1.1}$$

We then follow the idea of the proof of the Christoffel-Darboux identity for orthogonal ensembles. From the Bessel differential equation we have that

$$z^2 J_\alpha(z) = \alpha^2 J_\alpha(z) - zJ'_\alpha(z) - z^2 J''_\alpha(z). \tag{A.1.2}$$

This implies

$$\begin{aligned} 2a^2 K_\alpha^{\mathrm{Bes}}(a^2, b^2) &= \int_0^1 dt\, \frac{(at)^2}{t} J_\alpha(at)J_\alpha(bt) \\ &= \int_0^1 \frac{dt}{t}\left[\alpha^2 J_\alpha(at) - (at)J'_\alpha(at) - (at)^2 J''_\alpha(at)\right] J_\alpha(bt). \end{aligned} \tag{A.1.3}$$

Similarly,

$$2b^2 K_\alpha^{\mathrm{Bes}}(a^2, b^2) = \int_0^1 \frac{dt}{t}\left[\alpha^2 J_\alpha(bt) - (bt)J'_\alpha(bt) - (bt)^2 J''_\alpha(bt)\right] J_\alpha(at). \tag{A.1.4}$$

Combining the two, one gets

$$\begin{aligned} &2(a^2-b^2)K_\alpha^{\mathrm{Bes}}(a^2,b^2) \\ &= \int_0^1 dt\,\left[bJ'_\alpha(bt)J_\alpha(at) - aJ'_\alpha(at)J_\alpha(bt) + tb^2J''_\alpha(bt)J_\alpha(at) - ta^2J''_\alpha(at)J_\alpha(bt)\right]. \end{aligned} \tag{A.1.5}$$

By integration by parts on the two first terms of the integral, we get

$$\begin{aligned} 2(a^2-b^2)K_\alpha^{\mathrm{Bes}}(a^2,b^2) &= \left[t(bJ'_\alpha(bt)J_\alpha(at) - aJ'_\alpha(at)J_\alpha(bt))\right]_{t=0}^{t=1} \\ &= bJ'_\alpha(b)J_\alpha(a) - aJ'_\alpha(a)J_\alpha(b). \end{aligned} \tag{A.1.6}$$

Finally, one arrives at

$$K_\alpha^{\mathrm{Bes}}(a^2,b^2) = \frac{bJ'_\alpha(b)J_\alpha(a) - aJ'_\alpha(a)J_\alpha(b)}{2(a^2-b^2)}. \tag{A.1.7}$$

■

A.2 Jacobi's formula

The (i,j)-minor of a $n \times n$ matrix A, denoted M_{ij}, is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A. The adjugate of A is then defined as the transpose of the cofactor matrix C of A

$$\mathrm{adj}(A) := C^\top = \left((-1)^{i+j}M_{ji}\right)_{1\leqslant i,j\leqslant n}. \tag{A.2.1}$$

Proposition A.2.1 (Jacobi's formula) *For a $n \times n$ matrix $A(t)$,*

$$\frac{d}{dt}\det(A(t)) = \mathrm{Tr}\left(\mathrm{adj}(A(t))\,\frac{dA(t)}{dt}\right) = \det(A(t))\,\mathrm{Tr}\left(A(t)^{-1}\,\frac{dA(t)}{dt}\right) \tag{A.2.2}$$

The latter equality only holds if $A(t)$ is invertible. As a special case,

$$\frac{\partial\det(A)}{\partial A_{ij}} = \mathrm{adj}(A)_{ji}. \tag{A.2.3}$$

A proof of Proposition A.2.1 can be found in any linear algebra textbook such as [112] and consists simply in a cofactor expansion of the determinant.

One can apply Jacobi's formula to express a determinant of size $(n+1)\times(n+1)$ in terms of two determinants of size $n \times n$ via the differentiation of an auxiliary variable.

Proposition A.2.2 *Let a, $\{w_k\}_{k=1}^n$, $\{z_k\}_{k=1}^n$, $\{x_{jk}\}_{j,k=1}^n$ be complex numbers,*

$$\det\begin{bmatrix} a & w_k \\ z_j & x_{jk} \end{bmatrix}_{j,k=1}^n = a\det[x_{jk}]_{j,k=1}^n + \partial_t \det[x_{jk} - t\, z_j w_k]_{j,k=1}^n\Big|_{t=0} \tag{A.2.4}$$

Proof. Let us start by noticing that

$$\det\begin{bmatrix} a & w_k \\ z_j & x_{jk} \end{bmatrix}_{j,k=1}^n = a\det[x_{jk}]_{j,k=1}^n + \det\begin{bmatrix} 0 & w_k \\ z_j & x_{jk} \end{bmatrix}_{j,k=1}^n. \tag{A.2.5}$$

Focusing on the second term of the RHS and expanding first along the first column then along the first row yields

$$\det\begin{bmatrix} 0 & w_k \\ z_j & x_{jk} \end{bmatrix}_{j,k=1}^n = -\sum_{l,m=1}^n (-1)^{l+m} \det[x_{jk}]_{\substack{j\in[\![1,n]\!]\setminus\{l\} \\ k\in[\![1,n]\!]\setminus\{m\}}} z_l w_m. \tag{A.2.6}$$

On the other hand, using Jacobi's formula on the second term in (A.2.4) yields

$$\partial_t \det[x_{jk} - t\, z_j w_k]_{j,k=1} = \sum_{l,m=1}^n \underbrace{(-1)^{l+m} \det[x_{jk} - t\, z_j w_k]_{\substack{j\in[\![1,n]\!]\setminus\{l\} \\ k\in[\![1,n]\!]\setminus\{m\}}}}_{=(\mathrm{adj}(A(t)))_{ml}} \underbrace{(-z_l w_m)}_{=(\partial_t A(t))_{lm}}. \tag{A.2.7}$$

Setting $t = 0$ finishes the proof. ∎

A.3 Intersection between orthogonal polynomial ensembles and Pólya ensembles

A sufficient condition for an orthogonal polynomial ensemble to be also a Pólya ensemble is presented in the following lemma. This condition is indeed verified for Laguerre, Cauchy-Lorentz and Jacobi ensembles where the polynomial B is respectively proportional to $x \mapsto x$, $x \mapsto x(1+x)$ and $x \mapsto x(1-x)$.

Lemma A.3.1 *Let us consider an orthogonal polynomial ensemble (2.1.50) with weight ω. If for all $j = 0, \dots, n-1$,*

$$\omega(x)\mathcal{P}_j(x) = \frac{d^j}{dx^j}\left[B(x)^j \omega(x)\right], \tag{A.3.1}$$

with $\mathcal{P}_j$ some polynomial of degree j, and B a real polynomial of degree at most 2 with at least one real root $r_0 \in \mathbb{R}$, then the orthogonal ensemble is also a Pólya ensemble with Pólya weight given by

$$w_n(x) = \left(\frac{B(x+r_0)}{x}\right)^{n-1} \omega(x+r_0). \tag{A.3.2}$$

Proof. We first observe that, without loss of generality, r_0 can be taken to be zero as it is equivalent to redefining B and ω in (A.3.1) up to a shift. Thus one can write explicitly $B(x) = x(ax+b)$, $a, b \in \mathbb{R}$.

We will make use of the following identity [91, Eq. (4.18)]

$$\frac{d^j}{dx^j}\left[x^j f(x)\right] = \left[\prod_{k=1}^{j}(k + x\partial_x)\right] f(x), \tag{A.3.3}$$

which holds for a sufficiently differentiable function f and $j \in \mathbb{N}$. For $j = 0$, the product of the derivatives in the RHS of (A.3.3) is omitted.

For $j = n-1$, using (A.3.3), one has

$$\frac{d^{n-1}}{dx^{n-1}}\left[B(x)^{n-1}\omega(x)\right] = \frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1}w_n(x)\right] = \left[\prod_{k=1}^{n-1}(k + x\partial_x)\right] w_n(x), \tag{A.3.4}$$

where $w_n(x) = \left(\frac{B(x)}{x}\right)^{n-1}\omega(x)$.

To confirm that the orthogonal polynomial ensemble with weight ω is indeed a Pólya ensemble with Pólya weight $w_n(x) = (ax+b)^{n-1}\,\omega(x)$, one needs to show the following equality on the linear spans

$$\mathrm{Span}\left\{x^j\omega(x)\right\}_{j=0}^{n-1} = \mathrm{Span}\left\{(-x\partial_x)^j w_n(x)\right\}_{j=0}^{n-1}. \tag{A.3.5}$$

On the one hand, using (A.3.3) for w_n we have

$$\mathrm{Span}\left\{(-x\partial_x)^j w_n(x)\right\}_{j=0}^{n-1} = \mathrm{Span}\left\{\frac{d^j}{dx^j}\left[x^j w_n(x)\right]\right\}_{j=0}^{n-1}. \tag{A.3.6}$$

On the other hand, by assumption, we have

$$\mathrm{Span}\left\{x^j\omega(x)\right\}_{j=0}^{n-1} = \mathrm{Span}\left\{\frac{d^j}{dx^j}\left[B(x)^j\omega(x)\right]\right\}_{j=0}^{n-1}. \tag{A.3.7}$$

The equality between the RHS of (A.3.6) and (A.3.7) is then immediate as it is explicitly

$$\mathrm{Span}\left\{\frac{d^j}{dx^j}\left[x^j\,(ax+b)^{n-1}\,\omega(x)\right]\right\}_{j=0}^{n-1} = \mathrm{Span}\left\{\frac{d^j}{dx^j}\left[x^j(ax+b)^j\omega(x)\right]\right\}_{j=0}^{n-1}. \tag{A.3.8}$$

This finishes the proof. ■

List of abbreviations and symbols

SVD Singular value decomposition. 1

$\mathrm{U}(n)$ Group of unitary $n \times n$ matrices. 1

$\mathbb{R}_+^*$ The positive real line excluding 0: $\mathbb{R}_+ \setminus \{0\}$. 1

$X^\dagger$ Hermitian conjugate of the matrix X. $X^\dagger := \overline{(X^\top)}$. 1

$\mathrm{T}(n)$ Group of upper uni-triangular $n \times n$ matrices, i.e. upper triangular matrices whose diagonal is the identity matrix. 1

$\mathrm{GL}(n, \mathbb{C})$ Group of invertible $n \times n$ complex matrices. 2

f_{BU} Bi-unitarily invariant density on $\mathbb{C}^{n \times n}$. 4

$\mathrm{L}^1(B)$ Space of Lebesgue integrable functions on B. 4

$\mathbb{1}_n$ The identity matrix of size $n \times n$. 4

Θ Heaviside step function equal to 1 when the argument is a positive definite matrix and 0 otherwise. 4

LUE Complex Wishart/ Laguerre unitary ensemble. 8

χ**GUE** Chiral Gaussian unitary ensemble. 8

LOE Real Wishart/ Laguerre orthogonal ensemble. 8

χ**GOE** Chiral Gaussian orthogonal ensemble. 8

GinUE Ginibre unitary ensemble/ complex Ginibre ensemble. 8

GinOE Ginibre orthogonal ensemble/ real Ginibre ensemble. 8

GUE Gaussian unitary ensemble/ complex Gaussian ensemble. 8

GOE Gaussian orthogonal ensemble/ real Gaussian ensemble. 8

CUE Circular unitary ensemble. 10

JUE Jacobi unitary ensemble/ truncated unitary matrices. 10

Δ_n Vandermonde determinant of size $n \times n$. 10

RHS/LHS Right/Left hand side. 11

f_{SV} Any symmetric density on $\mathbb{R}_+^n$. In our context it refers to the underlying density on squared singular values of a bi-unitarily invariant ensemble. 13

DPP Determinantal point process. 13

K_n Kernel of a determinantal point process. 14

w_n Pólya weight function. 15

$\mathrm{C}^k(B)$ Space of k-times continuously differentiable functions on B. 15

$[\![\cdot]\!]$ Integer interval, e.g. $[\![1,n]\!] = \{1,\dots,n\}$. 16

f_{EV} Underlying density on the eigenvalues of a bi-unitarily invariant ensemble. 18

$\overline{z}$ Complex conjugate of z. 18

perm Permanent: totally symmetric version of the determinant. 18

S_n Finite symmetric group of permutation of n elements. 18

δ Dirac delta function. 21

$\sim$ **and** $\underset{x\to a}{\sim}$ Asymptotic equivalence. 26

$o(\cdot)$ Little o of the Landau notation. 26

$\mathcal{M}$ Mellin transform. 27

$\tilde{f}$ Mellin transform of f. 29

$\tilde{f}_x$ Incomplete Mellin transform of f. 29

$\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$ Space of symmetric Lebesgue integrable functions on $\mathbb{R}_+^n$. 29

$\mathcal{M}_{\mathrm{S}}$ Symmetrized multidimensional Mellin transform. 29

$\mathcal{S}$ Spherical transform. 30

$\mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n)$ Set of underlying densities on the eigenvalues of bi-unitarily invariant matrix ensembles. $\mathrm{L}^{1,\mathrm{EV}}(\mathbb{C}^n) = \mathcal{R}\mathrm{L}^{1,\mathrm{SV}}(\mathbb{R}_+^n)$. 31

$\mathcal{R}$ SEV transform. 31

$\mathrm{L}^{1,\mathrm{BU}}(\mathbb{C}^{n\times n})$ Set of bi-unitarily invariant density functions on $\mathbb{C}^{n\times n}$. 32

$\mathbb{E}[\cdot]$ Expected value on $\mathrm{GL}(n,\mathbb{C})$ with respect to the density f_{BU}. 33

$\mathcal{S}(B)$ Space of Schwartz functions on B. 34

$\mathrm{C_b}(B)$ Space of continuous bounded functions on B. 34

$\mu_{j,k}$ j,k-point correlation measure. Probability measure. 34

$f_{j,k}$ j,k-point correlation function. Probability density. 34

ρ_{EV} 1-point correlation/ level density on the squared eigenradii. 35

ρ_{SV} 1-point correlation/ level density on the squared singular values. 35

$\mathrm{cov}_{j,k}$ j,k-cross-covariance density function. 36

cov $1,1$-cross-covariance density function. 36

sgn Signum function. For $x \in \mathbb{R}$, $\operatorname{sgn}(x) = 1, 0, -1$, respectively, if $x > 0$, $x = 0$, $x < 0$. 60

$\mathrm{L}^1_{\mathrm{loc}}(B)$ Space of locally Lebesgue integrable functions on B, i.e. functions integrable on every compact subset $K \subseteq B$. 83

$O(\cdot)$ Big O of the Landau notation. 85

$\lceil \cdot \rceil$ Ceiling function $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geqslant x\}$. 89

erfc Complementary error function. $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. 114

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